Math 3110: Example Solving a Linear Diophantine Equation

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Example

The example below illustrates the process for solving the Diophantine equations $12x + 22y = 4$ (in other words, finding all integer solutions).

1. Compute the $gcd(22, 12)$ using the Euclidean algorithm:

$$\begin{aligned} 22 &= 1 \cdot 12 + 10 \\ 12 &= 1 \cdot 10 + 2 \\ 10 &= 5 \cdot 2 + 0 \end{aligned}$$

   You can skip this step if you are comfortable doing it in combination with the next step. But at this point we have learned that $\gcd(22, 12) = 2$ (from looking at the second-to-last line).

2. The corresponding Extended Euclidean Algorithm:

   Notation: The card

(x, y) gives c

   tells you that the linear combination $12x + 22y = c$.

   In what follows, notice that I choose $(0, 1)$ or $(1, 0)$ on the first two cards depending on the position of 12 and 22 in the equation. Just make sure your starting cards are true facts!

$(0, 1)$ gives 22	$= 1 \cdot$	$(1, 0)$ gives 12	$+$	$(-1, 1)$ gives 10
$(1, 0)$ gives 12	$= 2 \cdot$	$(-1, 1)$ gives 10	$+$	$(2, -1)$ gives 2
$(-1, 1)$ gives 10	$= 2 \cdot$	$(2, -1)$ gives 2	$+$	$(-11, 6)$ gives 0

3. Check that I didn't make arithmetic errors, by making sure the final card is correct:

$$12(-11) + 22(6) = 0.$$

4. Collect the fact from the second to last card on the right column:

$$12(2) + 22(-1) = 2.$$

5. To obtain a solution to our original problem, we multiply by an appropriate multiple (in this case, 2):
$$12(4) + 22(-2) = 4.$$
So $12x + 22y = 4$ has at least one solution, $(4, -2)$.

6. Find all solutions to the homogeneous linear Diophantine equations, i.e. the full set of solutions to $12x + 22y = 0$ is (by the theorem)
$$\left\{ \left(\frac{-22k}{\gcd(12,22)}, \frac{12k}{\gcd(12,22)} \right) : k \in \mathbb{Z} \right\} = \{(-11k, 6k) : k \in \mathbb{Z}\}.$$

7. The full set of solutions to $12x+22y = 4$ is obtained by combining the homogeneous solutions with the one particular solution we found:
$$= \{(4 - 11k, -2 + 6k) : k \in \mathbb{Z}\} = \{\ldots, (15, -8), (4, -2), (-7, 4), \ldots\}.$$