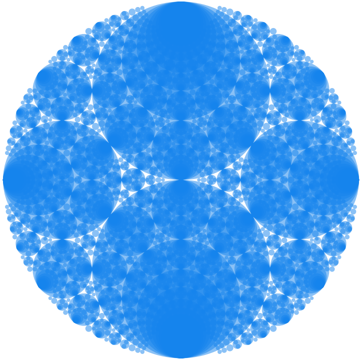
Reciprocity obstructions in Apollonian circle packings and continued fractions

Summer Haag, Clyde Kertzer, James Rickards, **Katherine E. Stange**



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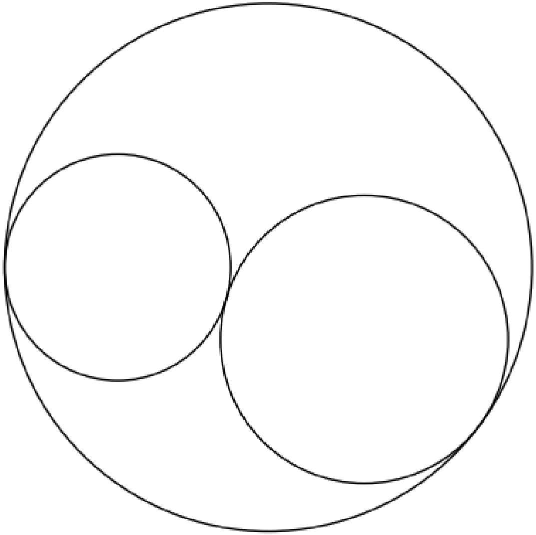
Apollonian circle packing



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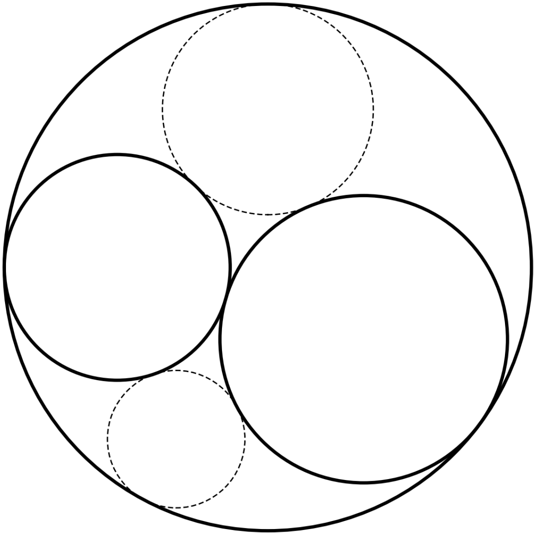

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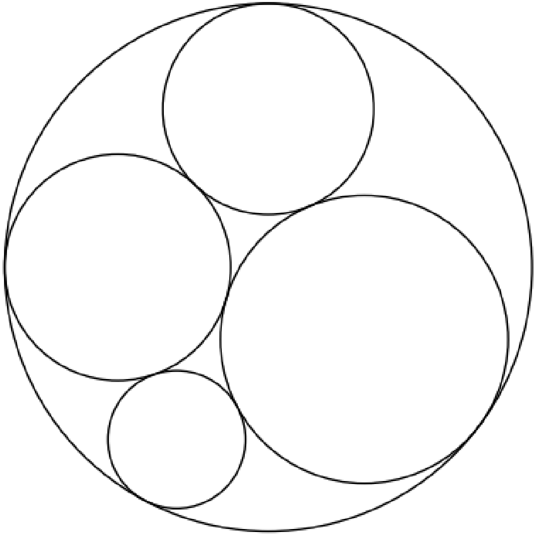

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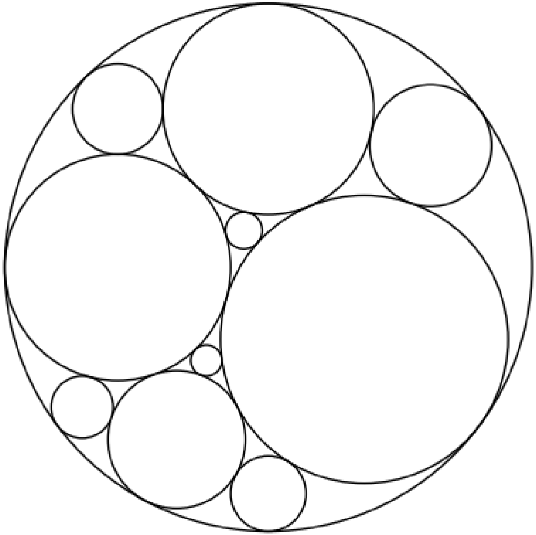

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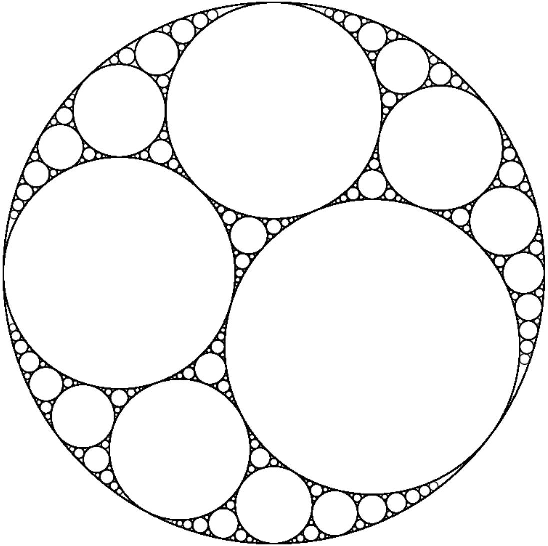

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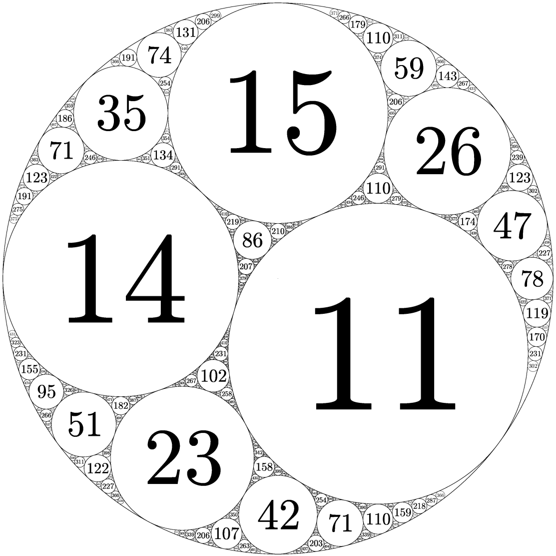

a theorem of Elizabeth and Descartes

Curvatures a, b, c, d of four mutually tangent circles (a *Descartes quadruple*) satisfy

$$2(a^2+b^2+c^2+d^2)=(a+b+c+d)^2.$$

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$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Given a, b, c, there are two possibilities d and d' satisfying

$$d + d' = 2(a + b + c).$$

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$$2(a^2+b^2+c^2+d^2)=(a+b+c+d)^2.$$

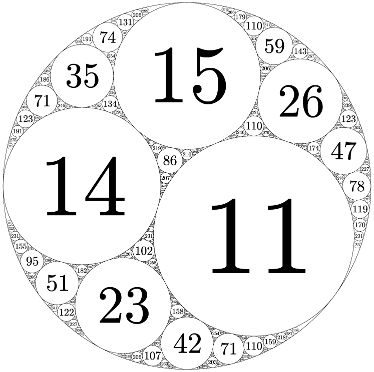
Given a, b, c, there are two possibilities d and d' satisfying

$$d+d'=2(a+b+c).$$

Therefore

$$a,b,c,d\in\mathbb{Z} \quad\Longrightarrow\quad \text{everything}\in\mathbb{Z}.$$

Curvatures

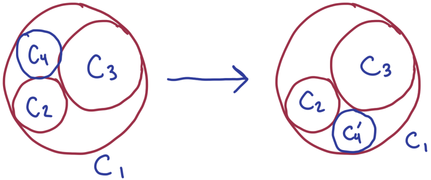


Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Apollonian group (Hirst)

Acting on Descartes quadruples of curvatures:



(4 swaps)

$$\mathscr{A} = \left\langle S_1, S_2, S_3, S_4 : S_i^2 = 1 \right\rangle < \mathrm{O}_Q(\mathbb{Z})$$

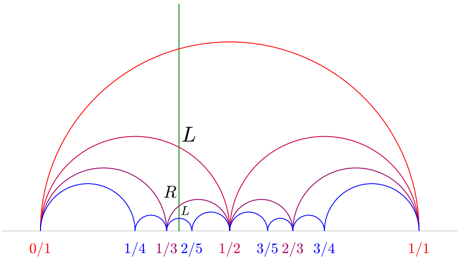
$\mathscr{A}$ is a *thin group*, meaning

- infinite index in $\mathrm{O}_Q(\mathbb{Z})$
- Zariski dense in O_Q

For example,

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

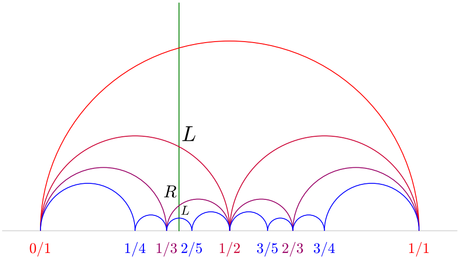
Continued Fractions (Series)



$$\begin{pmatrix}355\\113\end{pmatrix} = R^3L^7R^{15}L^1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}^3\begin{pmatrix}1&1\\0&1\end{pmatrix}^7\begin{pmatrix}1&0\\1&1\end{pmatrix}^{15}\begin{pmatrix}1&1\\0&1\end{pmatrix}^1\begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

Continued Fractions



Continued fraction semigroup:

$$\mathrm{SL}(2,\mathbb{Z})^{\geq 0} = \langle L, R\rangle^{+} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : a, b, c, d \geq 0 \right\}$$

Zaremba's Conjecture

Let $A \subseteq \mathbb{N}$ be a finite alphabet.

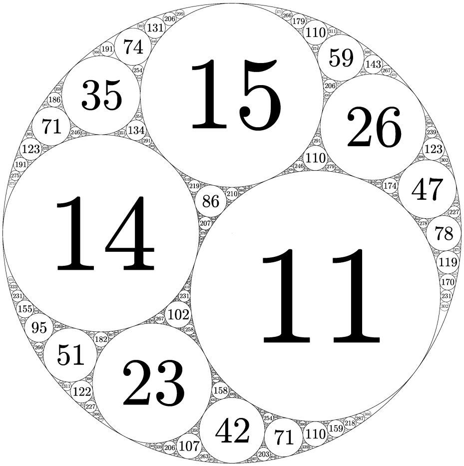
$$Z_A := \left\{ q \in \mathbb{N} : \begin{matrix} \exists p \in \mathbb{N}, \\ p/q \text{ has continued fraction expansion} \\ \text{using only coefficients from } A \end{matrix} \right\}$$

Conjecture (Zaremba)

Let $A = \{1,2,3,4,5\}$*. Then* $Z_A = \mathbb{N}$.

$$Z_A = \left\{ q \in \mathbb{Z} : \exists p \in \mathbb{N}, \begin{pmatrix} p \\ q \end{pmatrix} \in \Gamma_A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad \Gamma_A := \left\langle \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in A \right\rangle^+ .$$

Geometric considerations



Once $-6, 11, 14, 15$ are set, no room for $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, \ldots$

Analytic considerations

$$\#\{C : \mathrm{curv}(C) < X\} \sim c_A X^{1.30568\ldots}, \quad 1.30568\ldots = \text{Hausdorff dim.}$$

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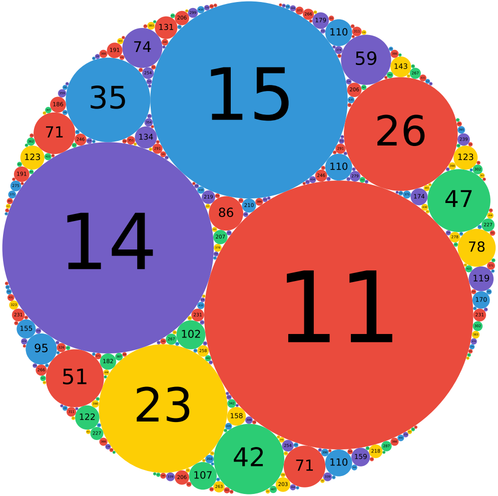
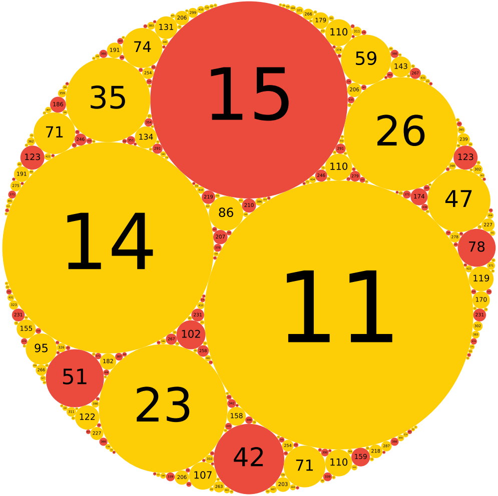
So that multiplicities, on average, are around $X^{0.3}$.

(Boyd, McMullen, Kontorovich-Oh)

Curvatures modulo 5



Curvatures modulo 3

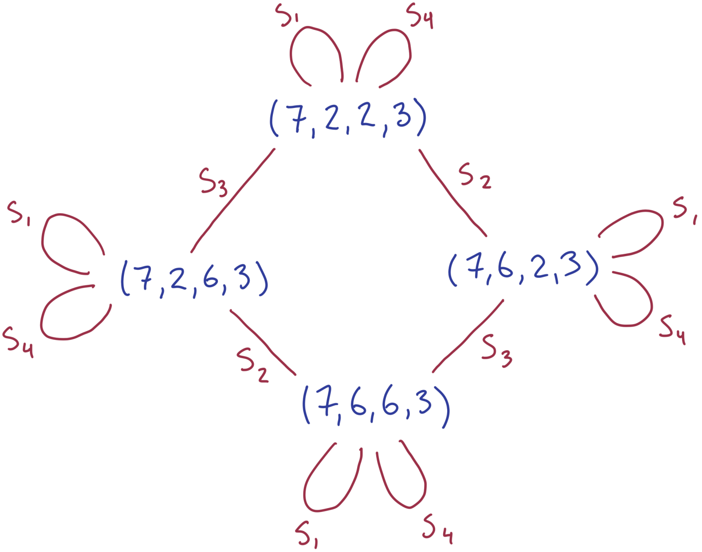


Algebraic considerations

Modulo 24, certain residue classes are missed.

type	allowed residues
$(6,1)$	$0,1,4,9,12,16$
$(6,5)$	$0,5,8,12,20,21$
$(6,13)$	$0,4,12,13,16,21$
$(6,17)$	$0,8,9,12,17,20$
$(8,7)$	$3,6,7,10,15,18,19,22$
$(8,11)$	$2,3,6,11,14,15,18,23$

Local obstruction: reduced Cayley graph



Apollonian group (Version 2)

Acting on Descartes quadruples of circles via Möbius transformations (swaps on a 'base quadruple' move the whole packing).

$$\mathscr{A} < \mathrm{PSL}_2(\mathbb{Z}[i])$$

There is an *exceptional isomorphism*

$$\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{SO}^+_{1,3}(\mathbb{R}).$$

Strong approximation (Fuchs)

Reduction modulo $\mathfrak{a}$

$$\mathrm{SL}_2(\mathbb{Z}[i]) \to \mathrm{SL}_2(\mathbb{Z}[i]/\mathfrak{a})$$

surjects for all $\mathfrak{a}$.

There are only finitely many primes $\mathfrak{p}$ where $\mathscr{A}/\mathfrak{p} \neq \mathrm{SL}_2/\mathfrak{p}$.

Where this fails, there exists an $m_{\mathfrak{p}}$ for which if $M \in \mathrm{SL}_2(\mathbb{Z}[i]/\mathfrak{p}^m)$, $m > m_{\mathfrak{p}}$ reduces mod $\mathfrak{p}^{m_{\mathfrak{p}}}$ to something in $\mathscr{A}/\mathfrak{p}^{m_{\mathfrak{p}}}$, then it reduces mod $\mathfrak{p}^m$ to something in $\mathscr{A}/\mathfrak{p}^m$.

Spectral gap (Varjú)

$A =$ adjacency matrix
$\mathbf{x} =$ vector of weights on vertices

Markov chain for flow of mass to neighbours:

$$\mathbf{x} \mapsto \frac{1}{4} A\mathbf{x}, \quad (x_w) \mapsto \left(\frac{1}{4} \sum_{w \sim v} x_w \right).$$

Eigenvalues: $1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq -1$
Spectral gap: $\lambda_0 > \lambda_1$ controls the rate of mixing

Expander family: Graphs G_i with $\epsilon := \limsup_{i \to \infty} \lambda_{G_i,1} < 1$

Possible thin group/semigroup orbit obstructions

1. **Definiteness**: Finite in one direction. (e.g., all positive)
2. **Thinness**: If growth rate is to slow to allow for increasing average multiplicity, can't expect density one.
3. **Congruence**: orbit $\cap \{a \pmod{m}\} = \emptyset$.
4. **Inherited**: Obstructions (such as Brauer-Manin) inherited from some larger algebraic set.

local-to-global

Conjecture (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden 2011):

In a primitive integral Apollonian circle packing,

- *curvatures satisfy a congruence condition modulo* 24, *and*
- *all sufficiently large integers satisfying this condition appear.*

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In other words,

$$\mathcal{K}(N) := \{n \leq N : n \text{ is a curvature }\} = kN + O(1),$$

Here $k = \dfrac{\#\text{ admissible curvatures modulo } 24}{24}$.

History

$$\mathscr{K}(N) := \{n \leq N : n \text{ is a curvature }\}$$

- Graham-Lagarias-Mallows-Wilks-Yan: $\mathscr{K}(N) \gg \sqrt{N}$.

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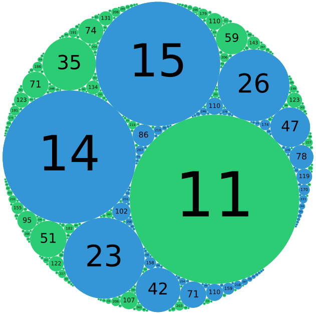
- Zhang, Fuchs-S.-Zhang: $\exists \eta > 0$, $\mathscr{K}(N) = kN + O(N^{1-\eta})$ for a larger class of packings.

Tool: quadratic forms (Sarnak, Graham-Lagarias-Mallows-Wilks-Yan)

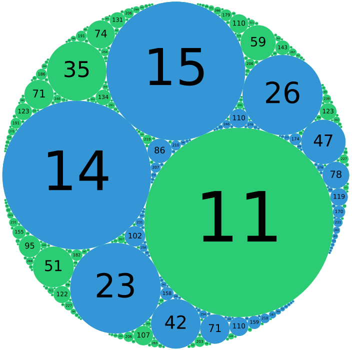


There is a bijection:

$$\left\{ \begin{array}{c} \text{curvatures of circles tangent} \\ \text{to fixed mother circle } C \text{ of curvature } a \end{array} \right\} \longleftrightarrow \{ f_C(x,y) - a : \gcd(x,y) = 1 \}$$

where f_C is a primitive integral binary quadratic form of discriminant $-4a^2$ associated to the ‘mother circle’.

Bourgain-Fuchs: $\mathscr{K}(N) \gg N$



$$S_C = \{\text{curvatures } \leq N \text{ represented by } f_C - a\}$$

Then

$$\mathscr{K}(N) \geq \sum_{C \in S} |S_C| - \sum_{C, C' \in S} |S_C \cap S_{C'}|$$

Bound the left by $\gg \eta N$ and the right by $\ll \eta^2 N$.

Bourgain-Kontorovich: $\mathscr{K}(N) = cN + O(N^{1-\eta})$

Write

$$\mathscr{R}_N(n) = \sum_{\gamma \in \mathscr{F}_T} \sum_{\substack{x,y \in \mathbb{Z}, \\ X \le x,y \le 2X}} 1_{f_\gamma(x,y)=n}$$

Here, $\mathscr{F}_T$ is a subset of $\mathscr{A}$ growing with T, and $N = T^2X^2$.

Goal: bound $\mathscr{R}_N(n) > 0$ for almost all n.

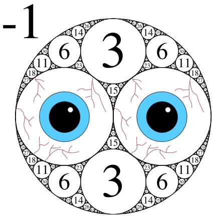
Tool: Hardy-Littlewood Circle Method.

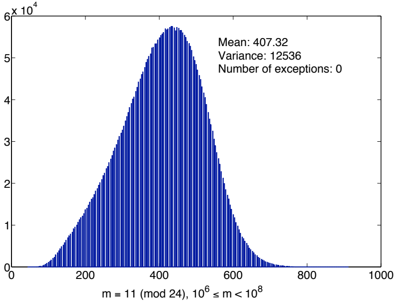
Computational Evidence (Fuchs-Sanden)

Computed curvatures up to:

$$10^8 \text{ for } (-1, 2, 2, 3)$$
$$5 \cdot 10^8 \text{ for } (-11, 21, 24, 28)$$

and observed that the multiplicity of a curvature was tending to increase.





Images from Fuchs, Sanden.

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Curvature c is *missing* in $\mathscr{A}$ if curvatures $\equiv c \pmod{24}$ appear in $\mathscr{A}$ but c does not.

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For $(-11,21,24,28)$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \pmod{24}$.

Summer 2023 REU

- Fix a pair of curvatures, and study what packings contain them.

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- Plot: for an admissible pair of residue classes modulo 24, black dot if no packing has that pair.
- Local-global: finitely many black dots on any row or column.

Typical graph

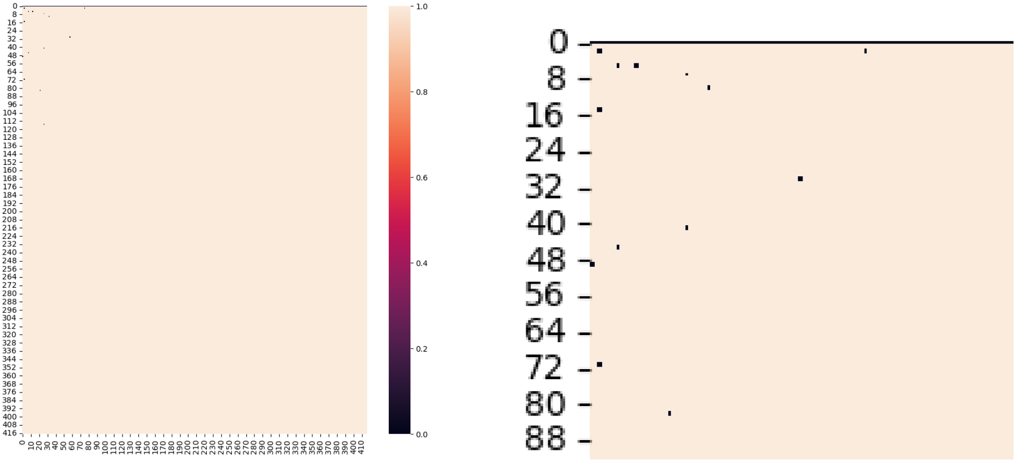


Figure: Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph

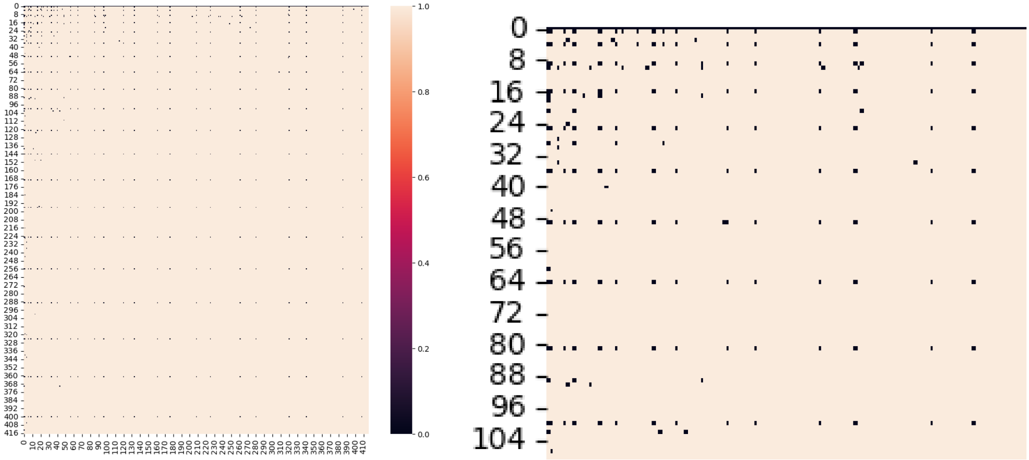
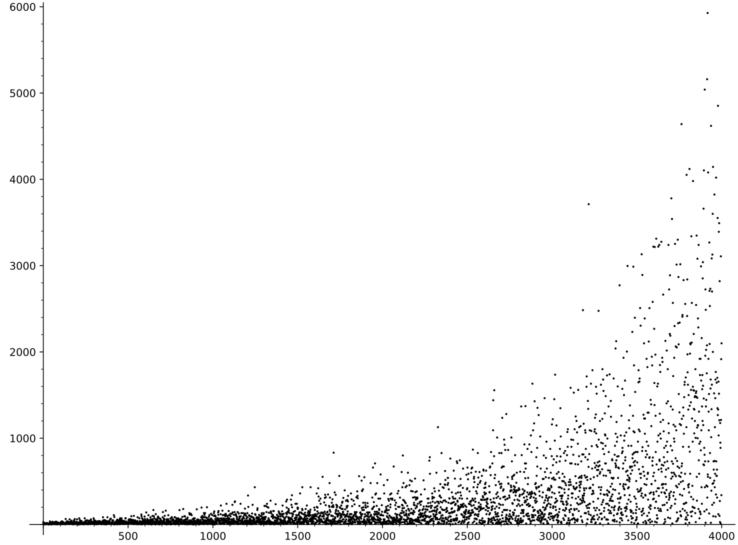


Figure: Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

differences between successive missing curvatures



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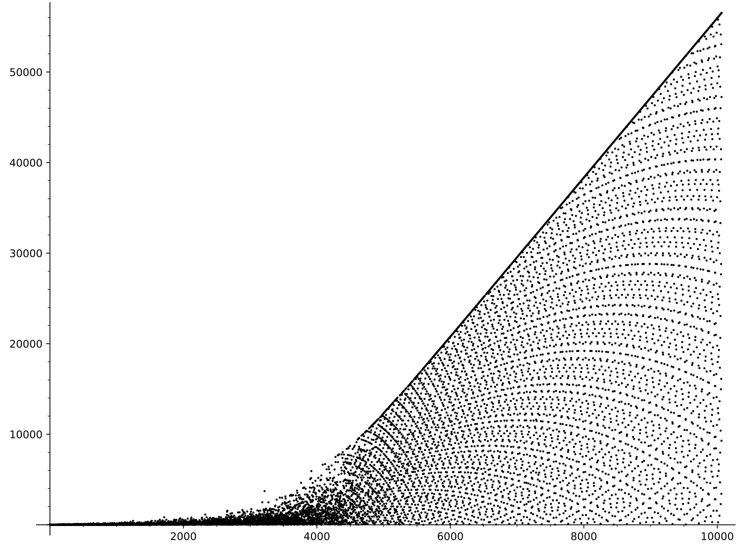

The conjecture is false

Theorem (Haag-Kertzer-Rickards-S.)

The Apollonian circle packing $\mathscr{A}$ generated by quadruple $(-3, 5, 8, 8)$ *has no square curvatures.*

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4. Define $\chi_2(\mathscr{C}) = 1$ or -1 according to above.
5. Note: $\chi_2(\mathscr{C}) = \left(\frac{a}{n}\right)$ for a curvature a coprime and tangent to $\mathscr{C}$.

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5. Note: $\chi_2(\mathscr{C})=\left(\frac{a}{n}\right)$ for a curvature a coprime and tangent to $\mathscr{C}$.
6. Observe: If $\chi_2(\mathscr{C})=-1$, then n is not square.

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2. Quadratic reciprocity:

$$\chi_2(\mathscr{C}_1)\chi_2(\mathscr{C}_2) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1 \quad \Longrightarrow \quad \chi_2(\mathscr{C}_1) = \chi_2(\mathscr{C}_2).$$

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3. Any two circles in $\mathscr{A}$ are connected by a path of pairwise coprime curvatures.
4. So $\chi_2(\mathscr{C})$ is independent of the choice of circle $\mathscr{C}$.

There are no squares in the packing

1. In base quadruple $(-3, 5, 8, 8)$, compute

$$\chi_2(\mathscr{A}) = \left(\frac{8}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

2. So no circle can have square curvature.

New invariants of a packing

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satisfies $\chi_4(\mathscr{C})^2 = \chi_2(\mathscr{C})$,
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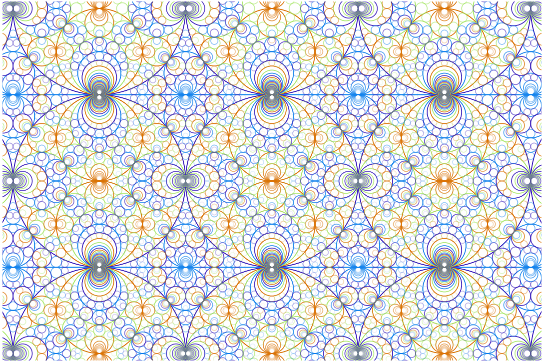
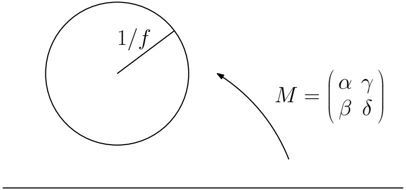
The values of χ_2 and χ_4 determine the quadratic and quartic obstructions respectively.

Quartic reciprocity





$$C \longrightarrow [\beta\mathbb{Z} + \delta\mathbb{Z}] \in \mathrm{Cl}(\mathcal{O}_f)$$

quadratic symbol → quartic symbol

value of form → element of lattice

Obstructions for types $(x, k, \chi_2(\mathscr{A}), \chi_4(\mathscr{A}))$

Type	Quadratic	Quartic	L-G false	L-G open
$(6, 1, 1, 1)$				$0, 1, 4, 9, 12, 16$
$(6, 1, 1, -1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0, 1, 4, 9, 12, 16$	
$(6, 1, -1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0, 1, 4, 9, 12, 16$	
$(6, 5, 1)$	$2n^2, 3n^2$		$0, 8, 12$	$5, 20, 21$
$(6, 5, -1)$	$n^2, 6n^2$		$0, 12$	$5, 8, 20, 21$
$(6, 13, 1)$	$2n^2, 6n^2$		0	$4, 12, 13, 16, 21$
$(6, 13, -1)$	$n^2, 3n^2$		$0, 4, 12, 16$	$13, 21$
$(6, 17, 1, 1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, 1, -1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, -1)$	$n^2, 2n^2$		$0, 8, 9, 12$	$17, 20$
$(8, 7, 1)$	$3n^2, 6n^2$		$3, 6$	$7, 10, 15, 18, 19, 22$
$(8, 7, -1)$	$2n^2$		18	$3, 6, 7, 10, 15, 19, 22$
$(8, 11, 1)$				$2, 3, 6, 11, 14, 15, 18, 23$
$(8, 11, -1)$	$2n^2, 3n^2, 6n^2$		$2, 3, 6, 18$	$11, 14, 15, 23$

New conjecture

Sporadic set $S_{\mathscr{A}}$ of AWOL curvatures: missing but not because of congruence or reciprocity obstuctions.

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Sporadic set $S_{\mathscr{A}}$ of AWOL curvatures: missing but not because of congruence or reciprocity obstuctions.

Conjecture (Haag-Kertzer-Rickards-S.)

Let $\mathscr{A}$ be a primitive Apollonian circle packing. Then $S_{\mathscr{A}}$ is finite.

Computational Evidence

- James wrote efficient C/PARI/GP code for missing curvatures (GitHub).
- computed many $S_{\mathscr{A}}(N)$
- 10^{10} in a few hours; 10^{12} possible

Possible thin group/semigroup orbit obstructions

1. **Definiteness**: Finite in one direction. (e.g., all positive)
2. **Thinness**: If growth rate is to slow to allow for increasing average multiplicity, can't expect density one.
3. **Congruence**: orbit $\cap \{a \pmod{m}\} = \emptyset$.
4. **Inherited**: Obstructions (such as Brauer-Manin) inherited from some larger algebraic set.
5. **Reciprocity**: Families ruled out by reciprocity laws.

What about Zaremba?

$$\#\{\text{ denominators} \le N\} \sim C_A N^{2\delta_A}, \quad \delta_A = \text{Hausdorff dimension.}$$

$$\delta_A > 1/2 \quad \Longrightarrow \quad \text{average multiplicity} \to \infty.$$

Conjecture (Hensley)

If $\delta_A > 1/2$, then D_A contains all but finitely many positive integers.

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If $\delta_A > 1/2$, then D_A contains all but finitely many positive integers.

- $A = \{1,2,3,4,5\}$: $\delta_A \approx 0.8368$;
- $A = \{1,2,3,4\}$: $\delta_A \approx 0.7889$;
- $A = \{1,2,3\}$: $\delta_A \approx 0.7057$;
- $A = \{1,2\}$: $\delta_A \approx 0.5313$;

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- $A = \{1,2,3\}$: $\delta_A \approx 0.7057$;
- $A = \{1,2\}$: $\delta_A \approx 0.5313$;
- $A = \{2,4,6,8,10\}$: $\delta_A \approx 0.5174$;
- Bourgain-Kontorovich: this last alphabet misses $3 \pmod 4$, disproving Hensley's conjecture.

Reciprocity obstructions in $\mathrm{SL}(2,\mathbb{Z})^{\geq 0}$

subsemigroups of $\mathrm{SL}(2,\mathbb{Z})^{\geq 0}$ $\longleftrightarrow$ restricted continued fraction expansions

A fascinating subset:

$$\Psi := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1^{\geq 0}(4) : \left(\frac{a}{b}\right) = 1 \right\}.$$

where

$$\Gamma_1^{\geq 0}(4) = \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z})^{\geq 0} : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod 4 \right\}.$$

Reciprocity obstructions in $\mathrm{SL}(2,\mathbb{Z})^{\geq 0}$

$$\Psi := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^{\geq 0}(4) : \left(\frac{a}{b}\right) = 1 \right\}.$$

Proposition

Ψ is a semigroup.

Reciprocity obstructions in $SL(2,\mathbb{Z})^{\geq 0}$

$$\Psi := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^{\geq 0}(4) : \left(\frac{a}{b}\right) = 1 \right\}.$$

Proposition

Ψ *is a semigroup.*

Fascinating consequence: If we say a rational p/q is "Kronecker positive" if $\left(\frac{p}{q}\right) = 1$, then this property is preserved under concatenation of continued fraction expansions (*).

Reciprocity obstructions in $\mathrm{SL}(2,\mathbb{Z})^{\geq 0}$

$$\Psi := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^{\geq 0}(4) : \left(\frac{a}{b}\right) = 1 \right\}.$$

Theorem (Rickards-S.)

Let x, y be positive coprime integers where y is odd and $\left(\frac{x}{y}\right) = -1$. Then the numerators and denominators of the orbit $\Psi \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ cannot be squares.

Once again, quadratic reciprocity.

Reciprocity obstructions in thin semigroups

$$\Psi_1 := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right\rangle^+ .$$

- *thin*, $\delta_1 \approx 1.4386$
- congruence obstructions modulo 4

Theorem (Rickards-S.)

Numerators of $\Psi_1 \cdot \binom{2}{3}$ have no congruence obstructions, yet cannot be square.

Conjecture

Every non-square integer $n > 10569$ occurs as a numerator in $\Psi_1 \cdot \binom{2}{3}$.

(data up to 10^7)

In terms of continued fractions

Theorem (Rickards-S.)

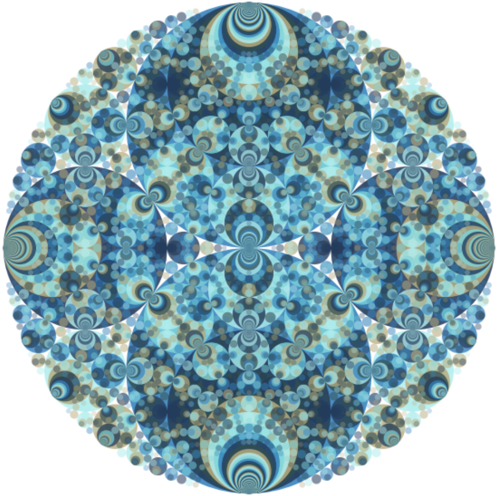
Let

$$S = \left\{ \frac{p}{q} = [0; a_1, a_2, \ldots, a_n, 1, 1, 2] : a_i \in \{4, 8, 12, \ldots, 128\} \right\}.$$

Then the limit set has Hausdorff dimension $> 1/2$*, no congruence obstruction to squares, but no square denominators.*

Disproves a conjecture of Bourgain and Kontorovich.

Thank you



Tables

Packing	Type	N	$\lvert S_{\mathscr{A}}(N)\rvert$	$\max(S_{\mathscr{A}}(N))$	$\approx \frac{N}{\max(S_{\mathscr{A}}(N))}$
$(0,0,1,1)$	$(6,1,1,1)$	10^{10}	215	1199820	8334.58
$(-12,16,49,49)$		10^{11}	275276	5542869468	18.04
$(-20,36,49,49)$		10^{12}	2014815	55912619880	17.89
$(-8,12,25,25)$	$(6,1,1,-1)$	10^{10}	47070	517280220	19.33
$(-12,25,25,28)$		10^{11}	238268	5919707820	16.89
$(-15,24,40,49)$		$2\cdot 10^{11}$	639149	12692531688	15.75
$(-15,28,33,40)$	$(6,1,-1)$	10^{10}	80472	820523160	12.19
$(-20,33,52,57)$		10^{11}	240230	4127189100	24.23
$(-23,40,57,60)$		10^{11}	392800	8689511520	11.51
$(-4,5,20,21)$	$(6,5,1)$	10^{10}	3659	32084460	311.68
$(-16,29,36,45)$		10^{10}	80256	927211800	10.79
$(-19,36,44,45)$		10^{11}	177902	3603790320	27.75
$(-3,5,8,8)$	$(6,5,-1)$	10^{10}	676	3122880	3202.17
$(-12,21,29,32)$		10^{10}	30347	312225420	32.03
$(-19,32,48,53)$		$2.5\cdot 10^{10}$	168264	2286209460	10.94

Tables

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$(-3,4,12,13)$	$(6,13,1)$	10^{10}	731	7354464	1359.72
$(-12,21,28,37)$		10^{11}	234386	3470731680	28.81
$(-11,16,36,37)$		10^{10}	20748	226988340	44.06
$(-8,13,21,24)$	$(6,13,-1)$	10^{10}	5273	45348900	220.51
$(-11,21,24,28)$		10^{10}	21003	176441136	56.68
$(-20,37,45,52)$		10^{11}	229356	4079861484	24.51
$(-16,32,33,41)$	$(6,17,1,1)$	10^{10}	81777	841440840	11.88
$(-7,8,56,57)$		10^{10}	55057	595231740	16.80
$(-16,20,81,81)$		10^{12}	1075024	26983035480	37.06
$(-4,8,9,9)$	$(6,17,1,-1)$	10^{10}	2057	10742460	930.89
$(-7,9,32,32)$		10^{10}	34916	367956840	27.18
$(-15,32,32,33)$		10^{11}	585942	8505627180	11.76
$(-7,12,17,20)$	$(6,17,-1)$	10^{10}	3744	17141220	583.39
$(-12,17,41,44)$		10^{10}	31851	270186456	37.01
$(-15,24,41,44)$		10^{10}	80106	803343900	12.45

Tables

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$(-5,7,18,18)$	$(8,7,1)$	10^{10}	16417	86709570	115.33
$(-6,10,15,19)$		10^{10}	24305	133977255	74.64
$(-9,18,19,22)$		10^{10}	14866	82815750	120.75
$(-2,3,6,7)$	$(8,7,-1)$	10^{10}	236	429039	23307.90
$(-5,6,30,31)$		10^{10}	19695	97583070	102.48
$(-14,27,31,34)$		$2\cdot 10^{10}$	99294	1643827935	12.17
$(-1,2,2,3)$	$(8,11,1)$	10^{10}	61	97287	102788.66
$(-9,14,26,27)$		10^{10}	17949	85926675	116.38
$(-10,18,23,27)$		10^{10}	25944	124625694	80.24
$(-6,11,14,15)$	$(8,11,-1)$	10^{10}	3381	20149335	496.29
$(-10,14,35,39)$		$4\cdot 10^{10}$	256228	2934238515	13.63
$(-13,23,30,38)$		10^{10}	71341	598107510	16.72