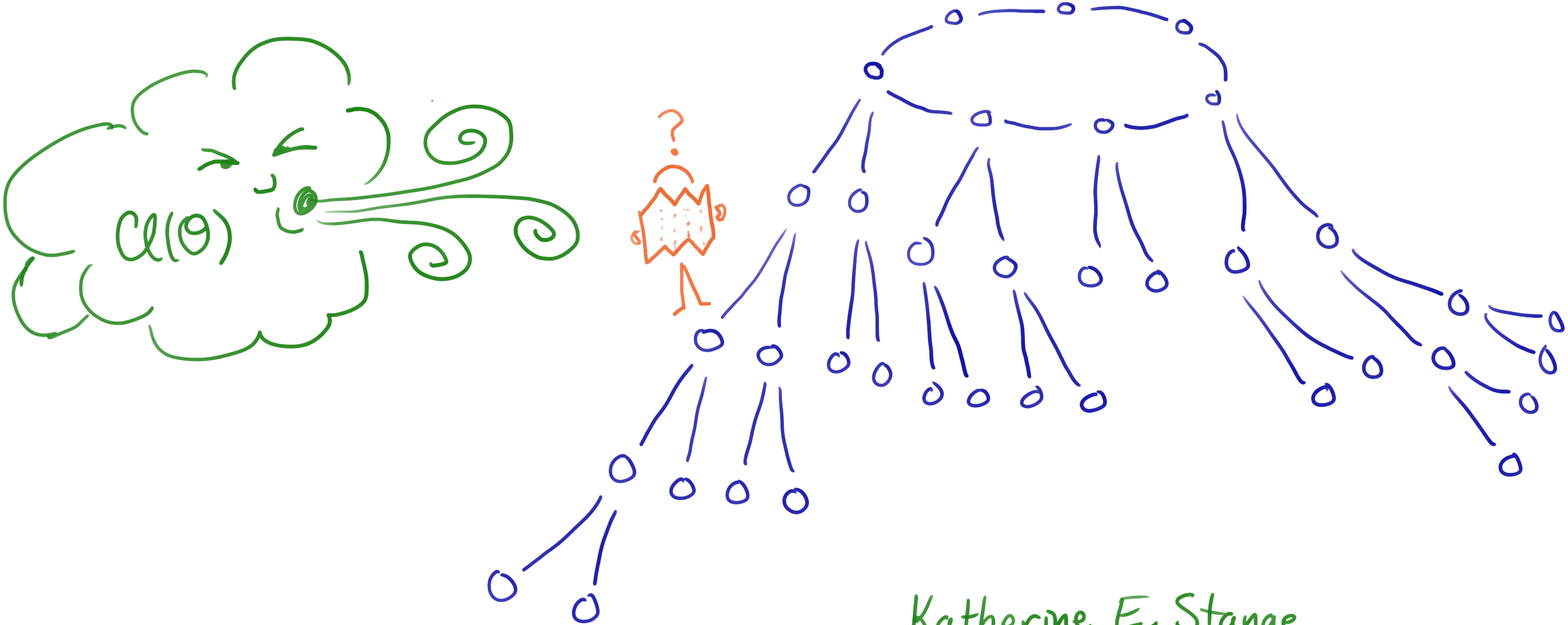
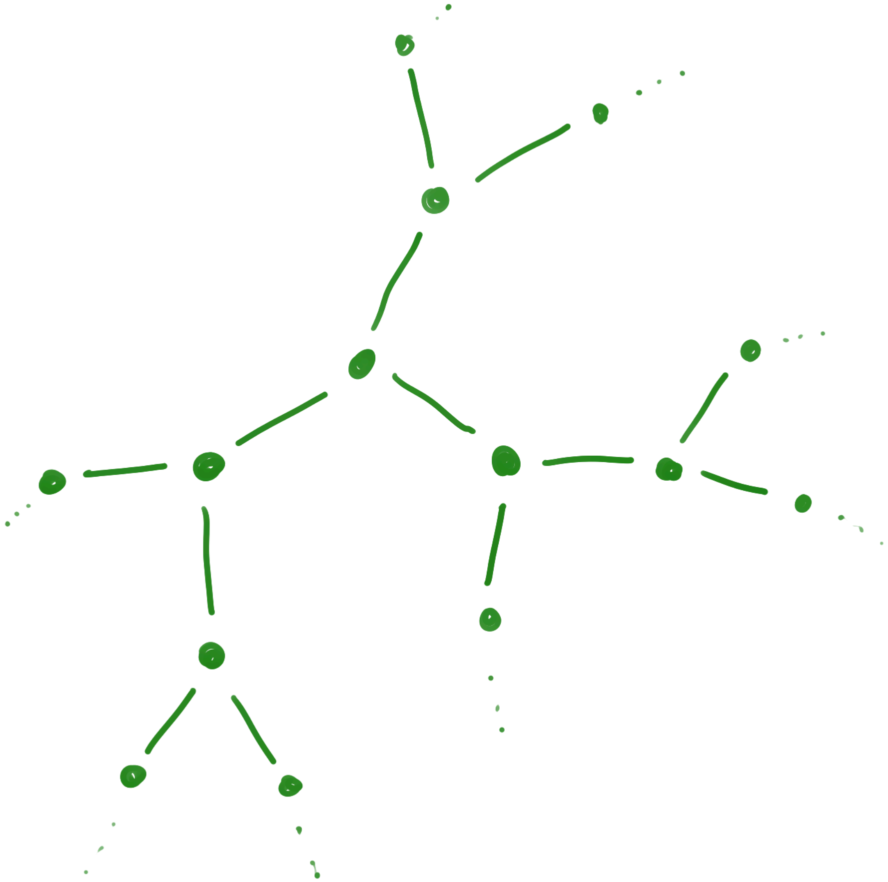
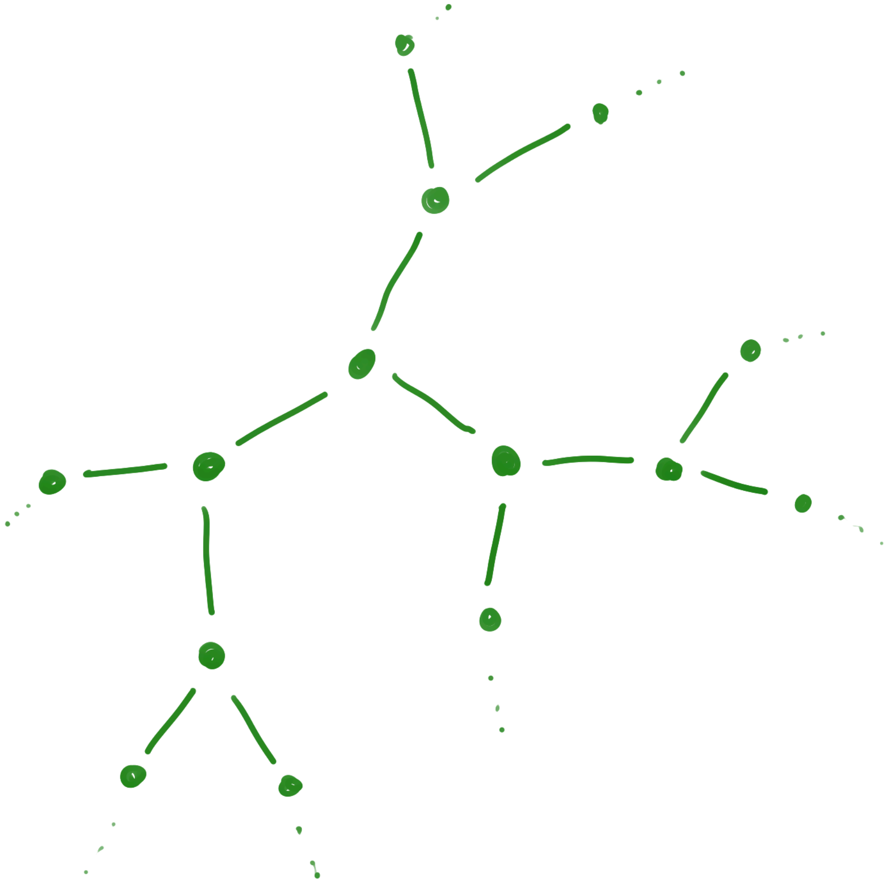
Orienteering on isogeny volcanoes

joint with Sarah Arpin, Mingjie Chen,
Kristin E. Lauter, Renate Scheidler,
Ha T. N. Tran



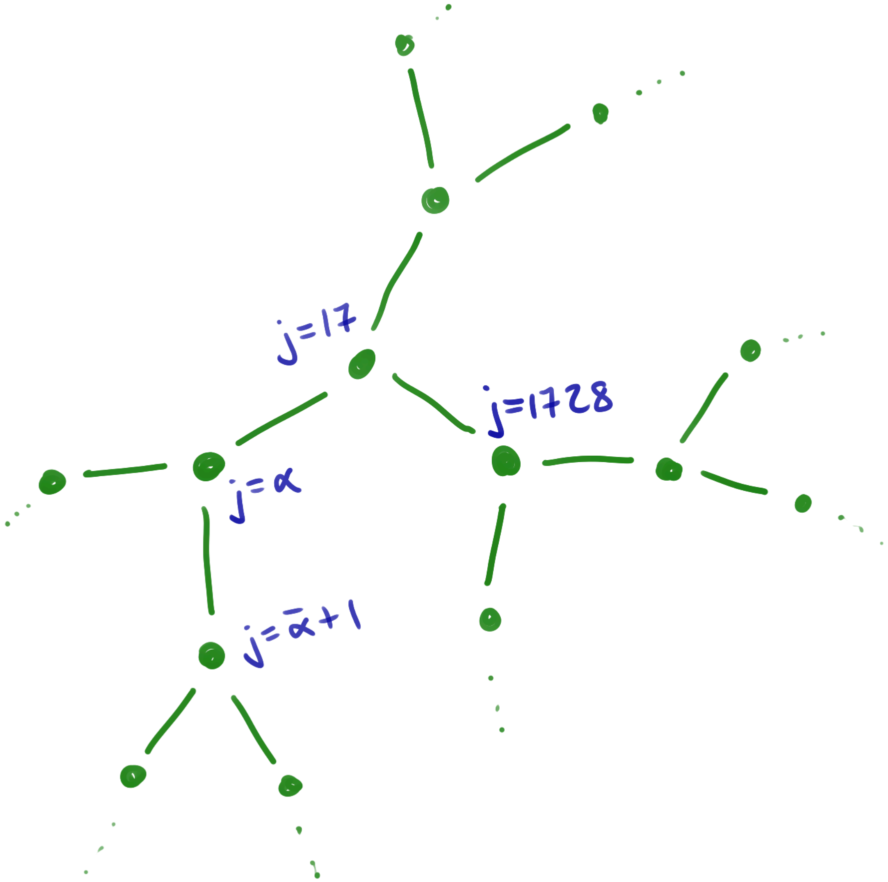
Katherine E. Stange
University of Colorado Boulder





ℓ-isogeny graph

ℓ = small prime



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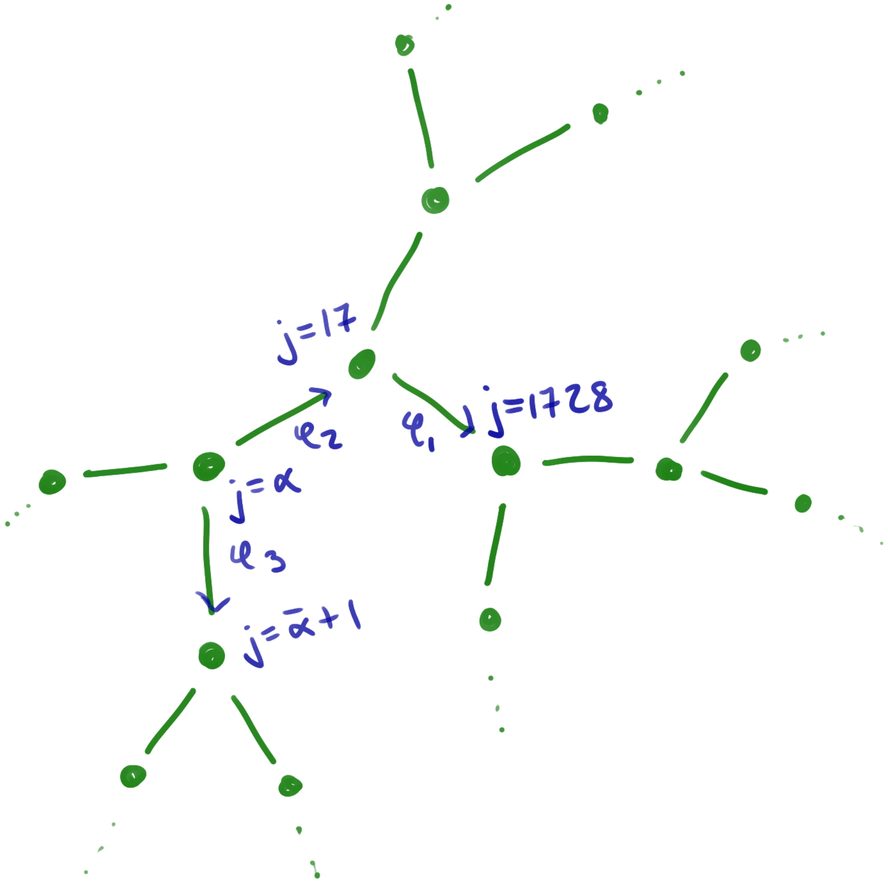
ℓ = small prime
p = large prime

vertices = elliptic curves $/\overline{\mathbb{F}}_p$ up to isom.



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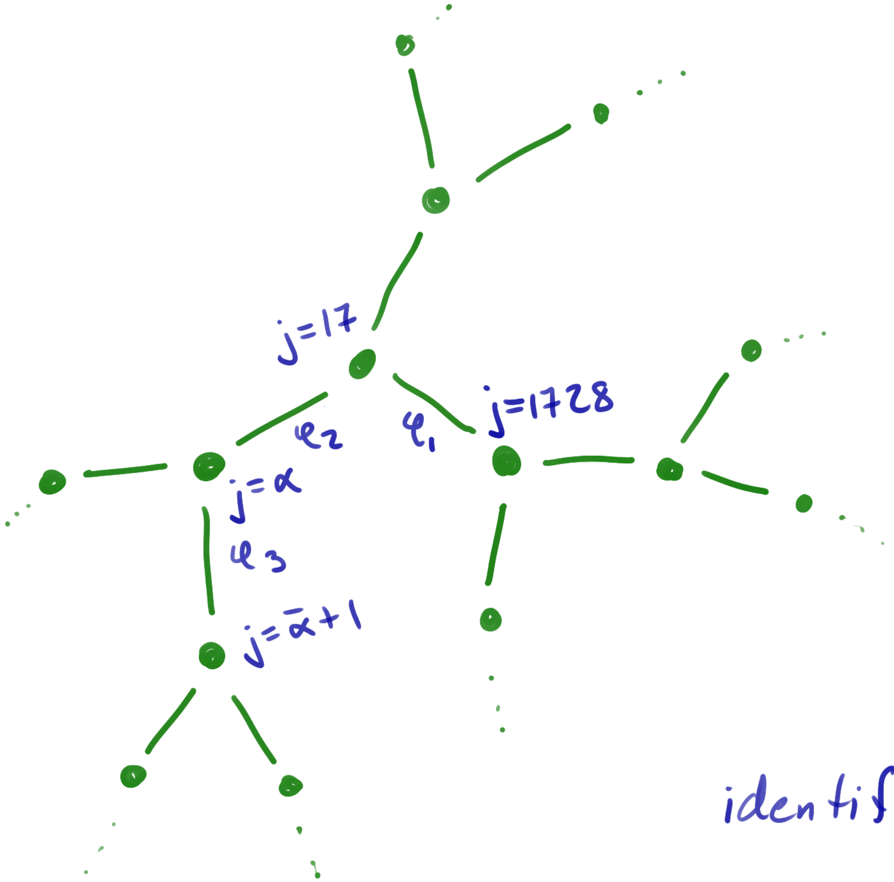

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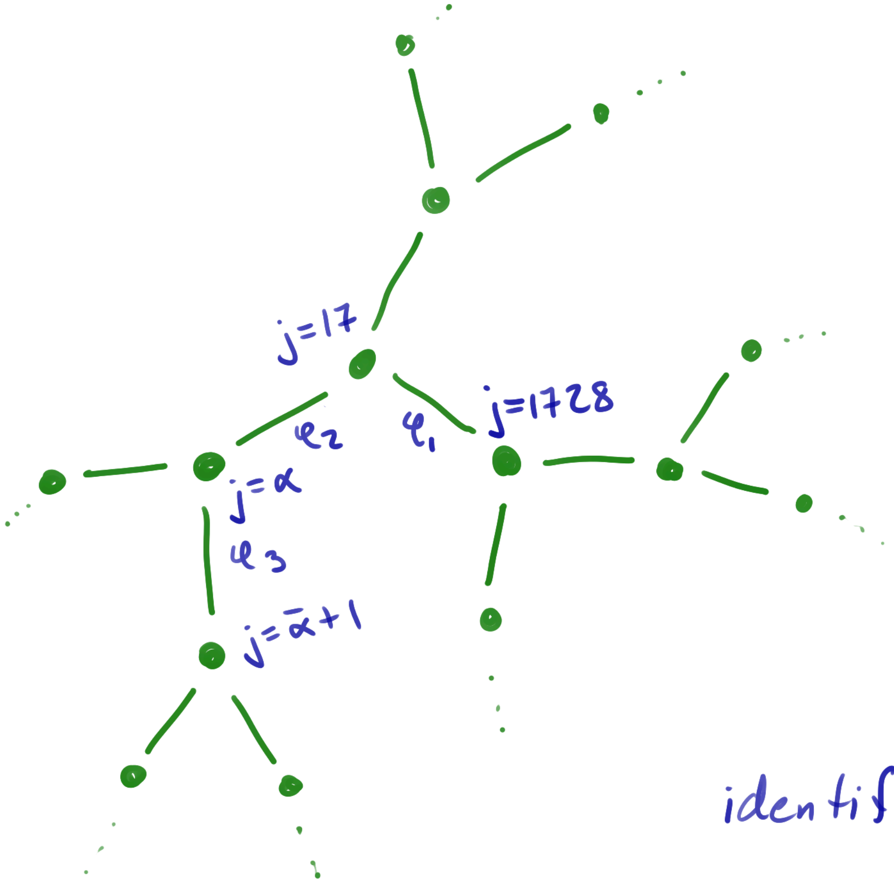
identifying φ with $\hat{\varphi}$ gives an undirected $(\ell+1)$-regular graph

$$\begin{tikzcd}\bullet \arrow[r, shift left, "\varphi"] & \bullet \arrow[l, shift left, "\hat{\varphi}"]\end{tikzcd}$$



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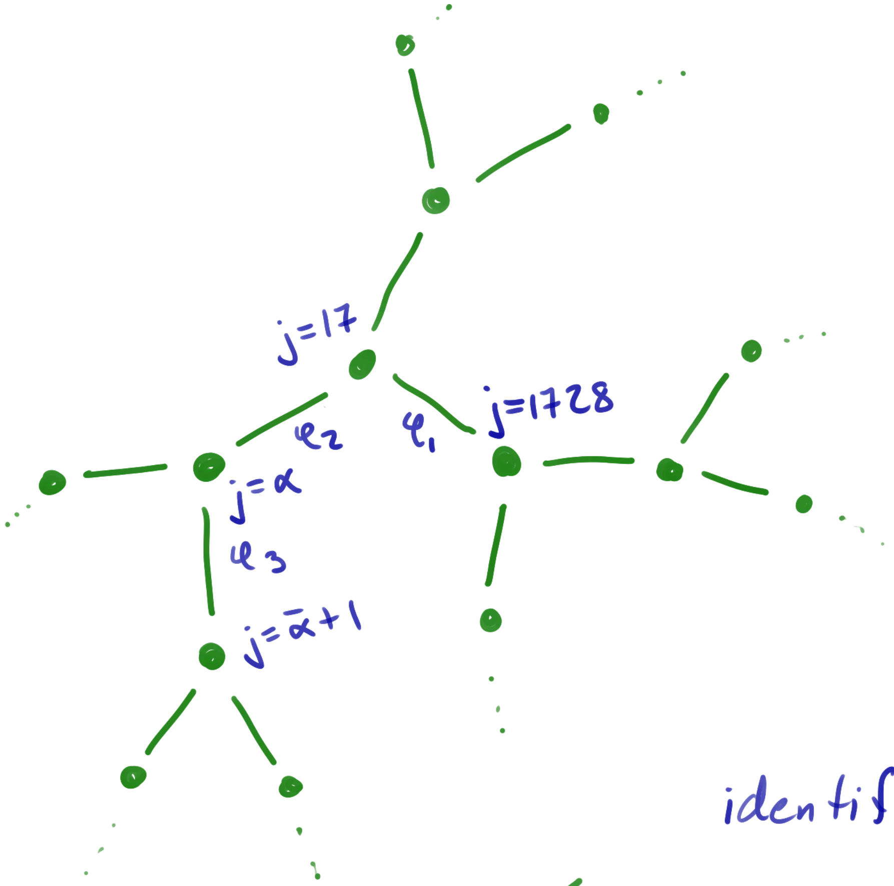

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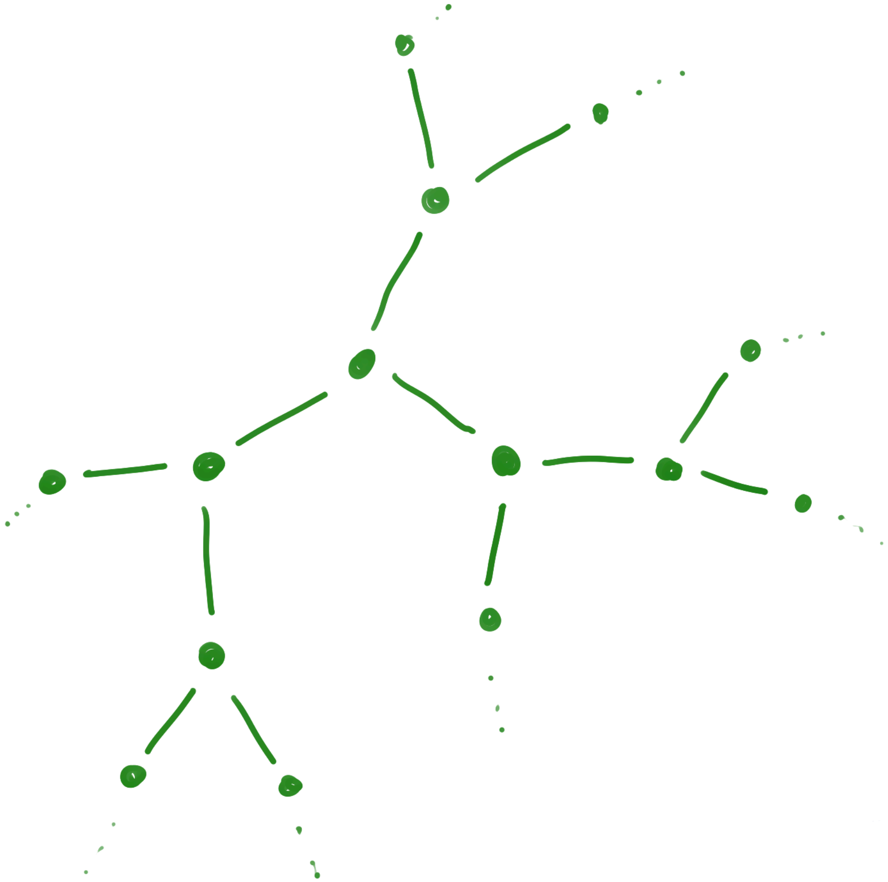
End(E)

imag. quad. order
or
quaternion order



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K = quadratic imag. # fld

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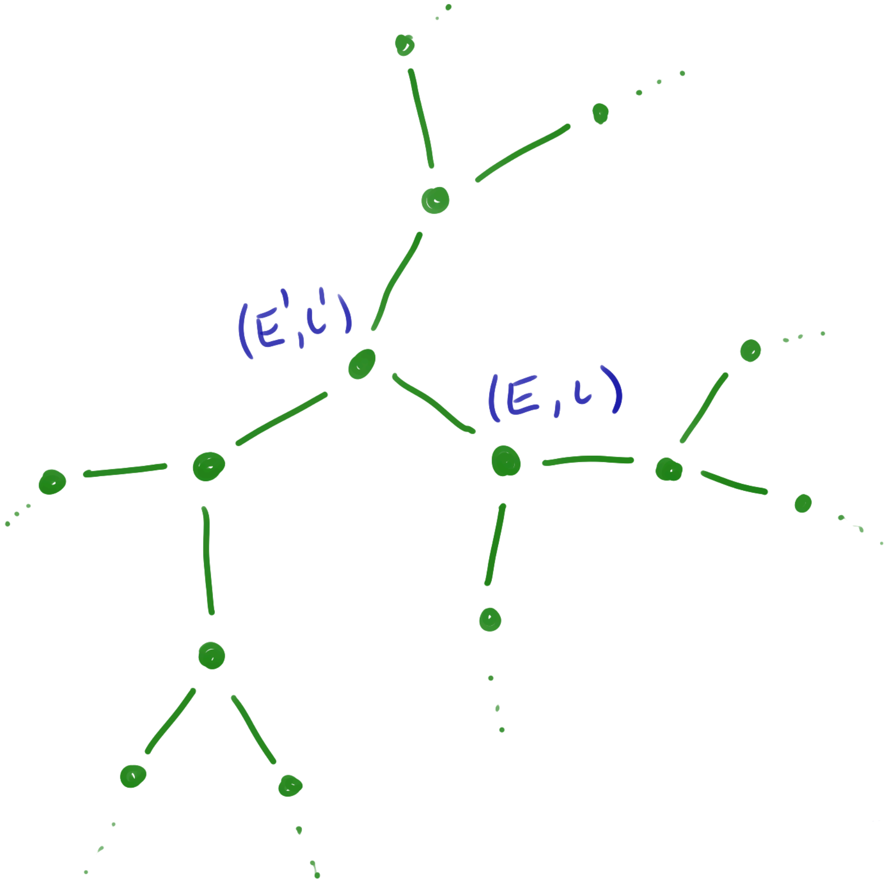

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vertices = elliptic curves $/\overline{\mathbb{F}_p}$ up to isom. together with a K-orientation

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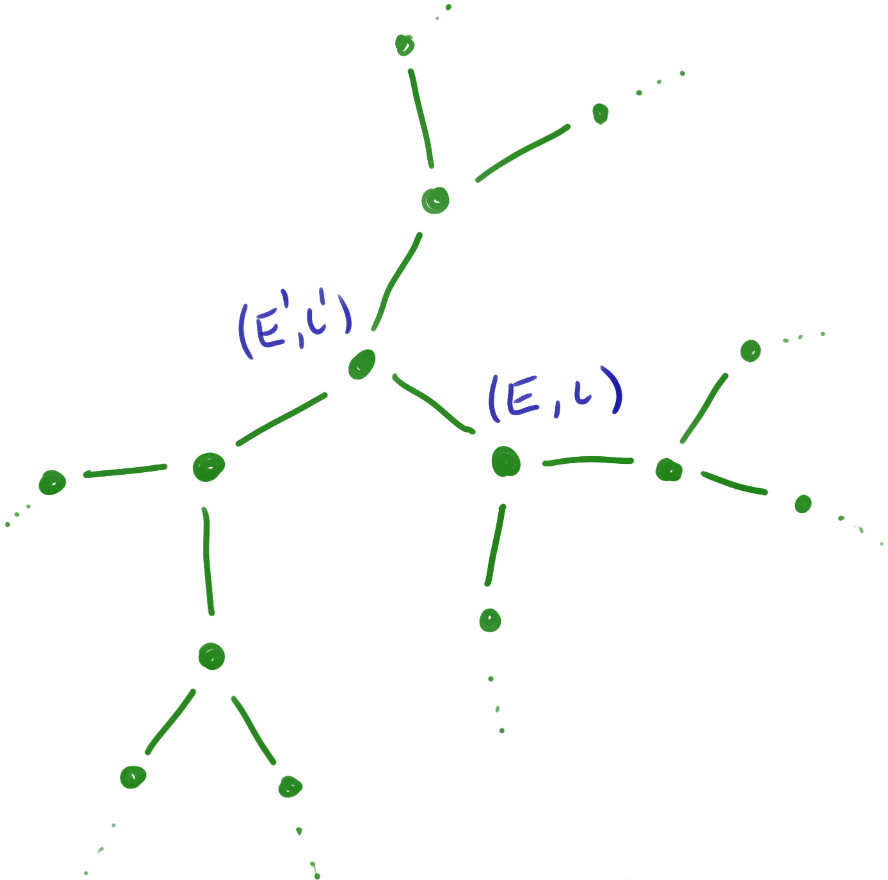

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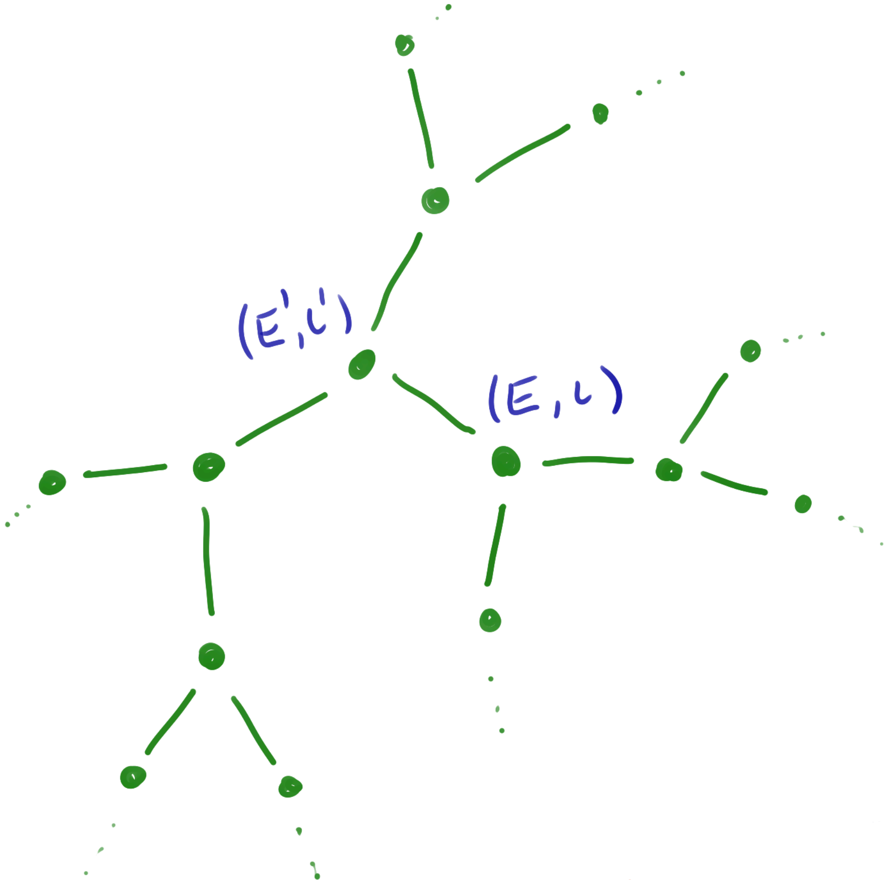
$\iota : K \hookrightarrow \mathrm{End}(E) \otimes \mathbb{Q}$



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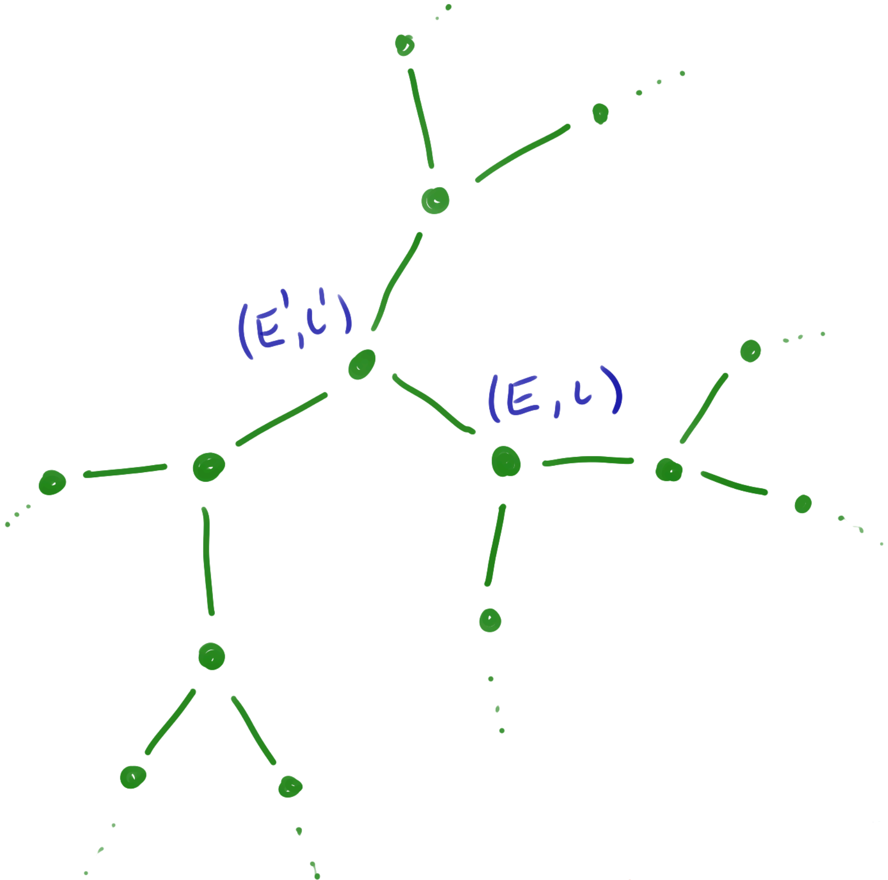

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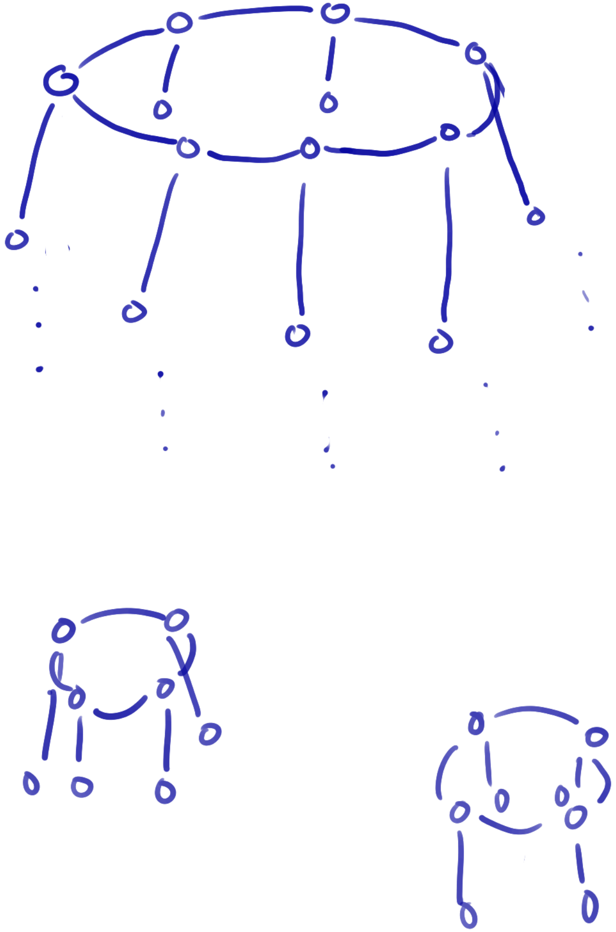
$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "\iota(\alpha)"] & E' \arrow[d, "\iota'(\alpha)"] \\
E \arrow[r, "\varphi"] & E'
\end{tikzcd}$$

$$\iota'(\alpha) = \varphi\, \iota(\alpha)\, \hat{\varphi}$$

ℓ-isogeny graphs

ordinary



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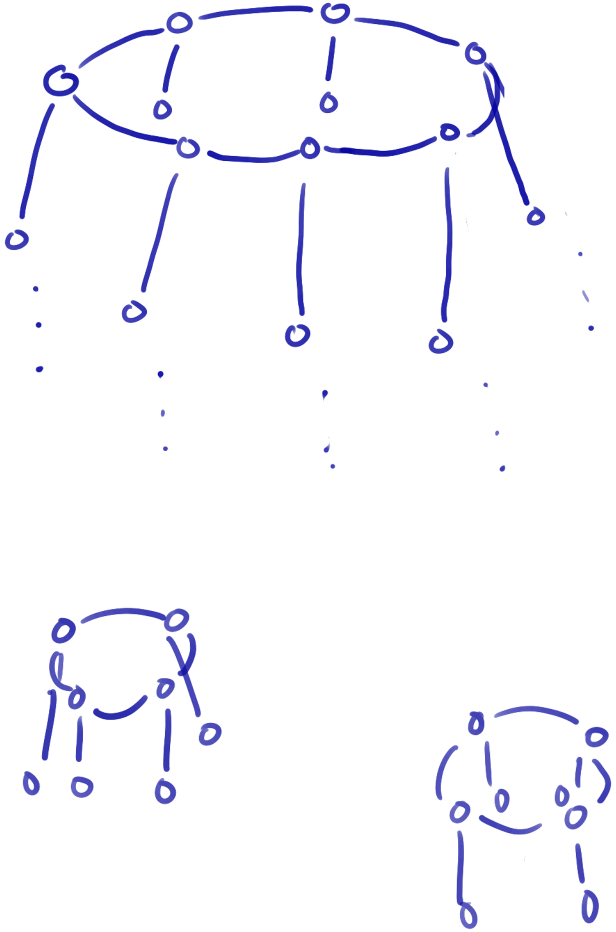
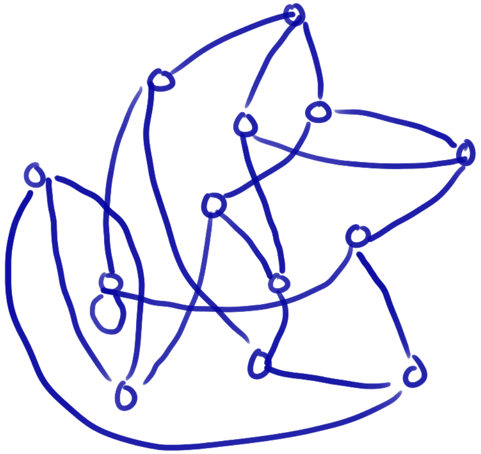


supersingular



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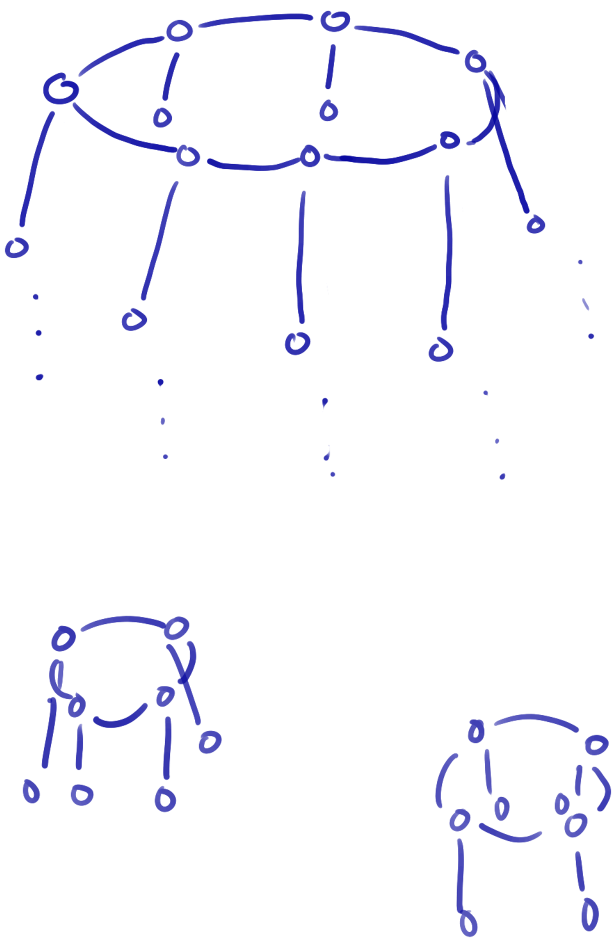

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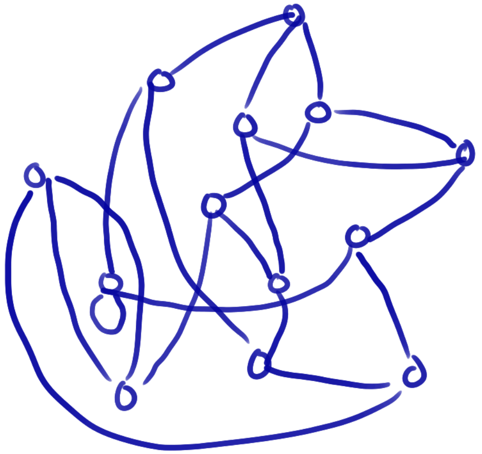

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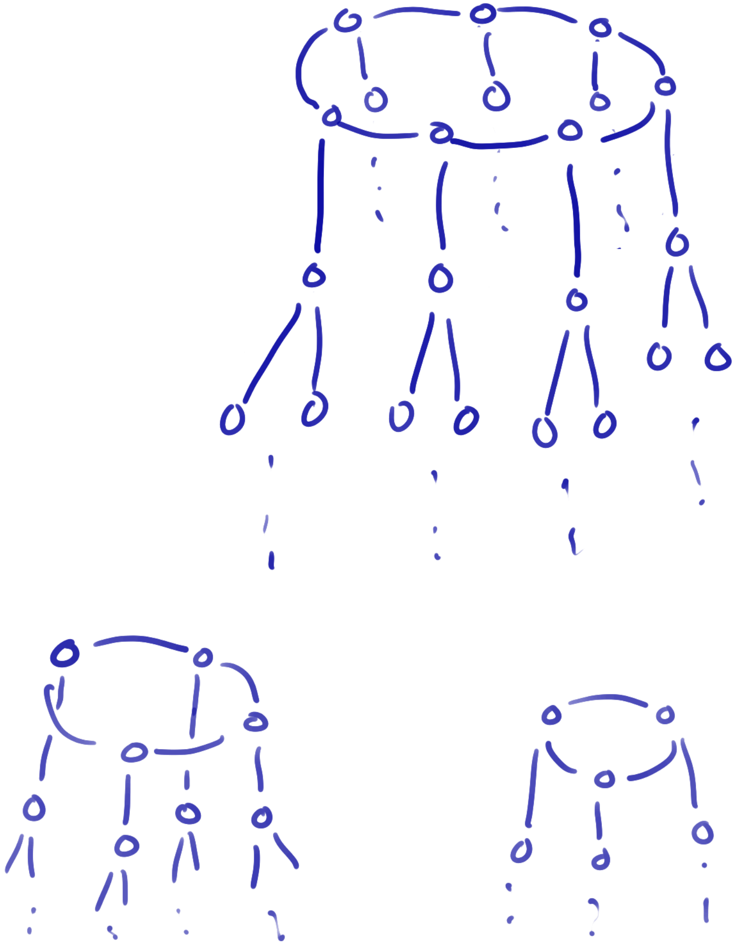

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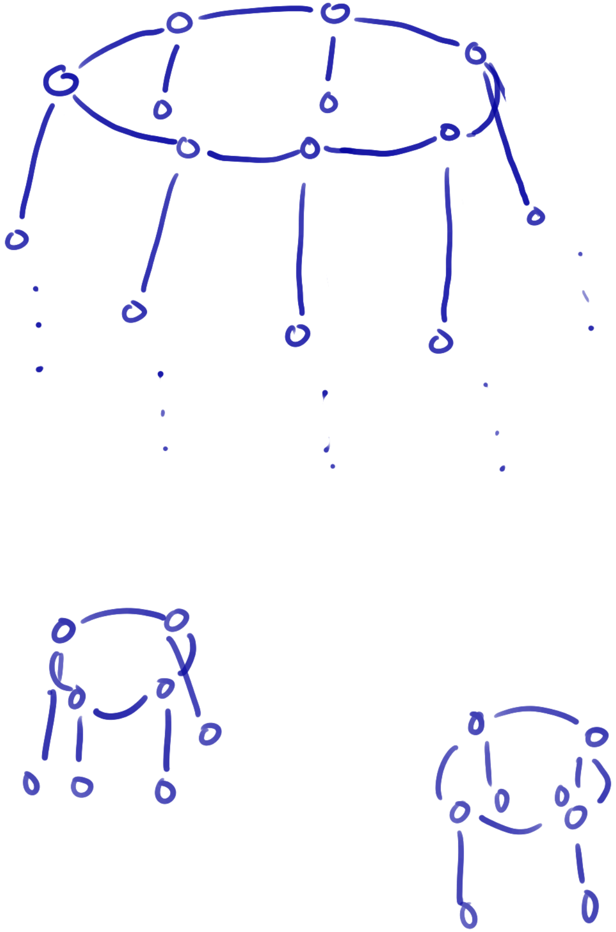

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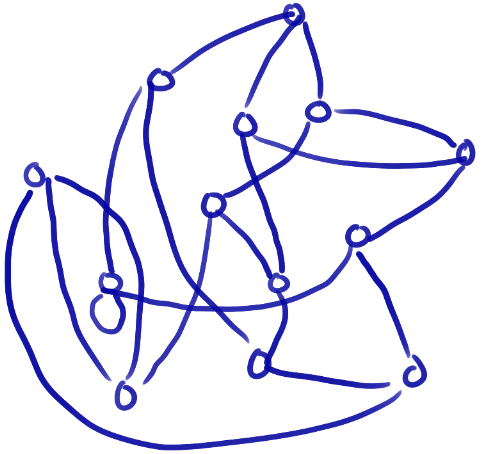

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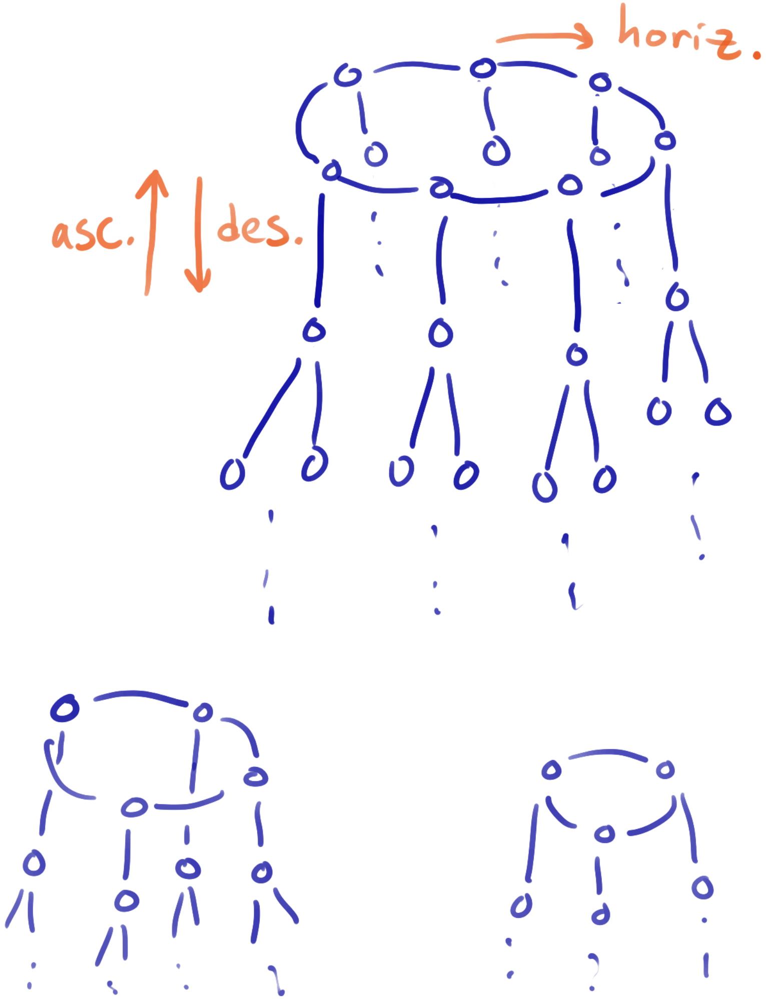

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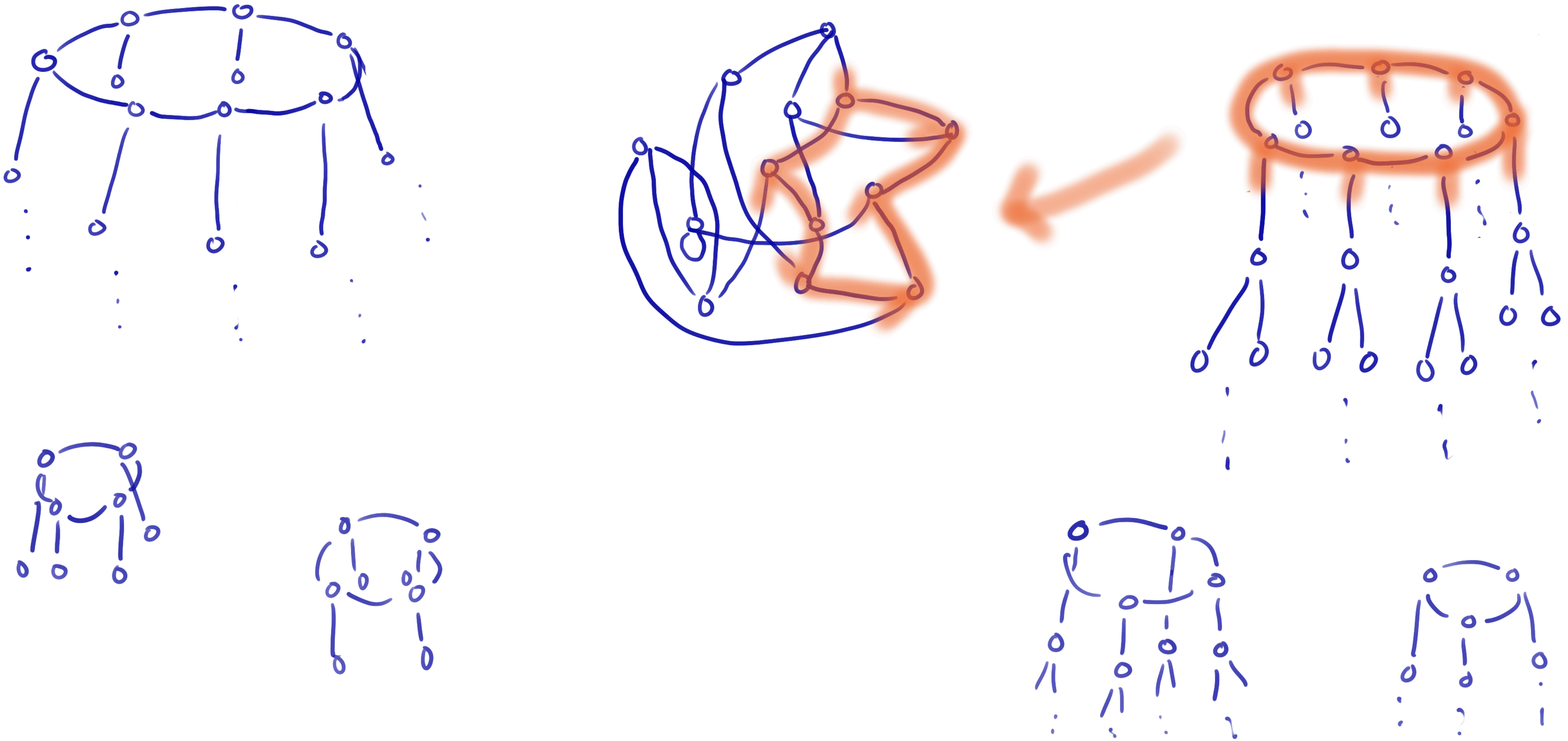

$SS_{\Theta}^{pr}(p) = \{ (E, \iota) : \iota \text{ is } \Theta\text{-primitive} \}$

quadratic order

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$$E[\mathfrak{a}] := \bigcap_{\alpha \in \iota(\mathfrak{a})} \ker(\alpha) \quad \text{for } \mathfrak{a} \subseteq \mathcal{O} \text{ } \overset{\text{inv.}}{\text{ideal}}$$

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$$[\mathfrak{a}]\cdot E = E/E[\mathfrak{a}]$$

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$Cl(\mathcal{O})$ acts freely on $SS^{pr}_{\mathcal{O}}(p)$

with 1 or 2 orbits

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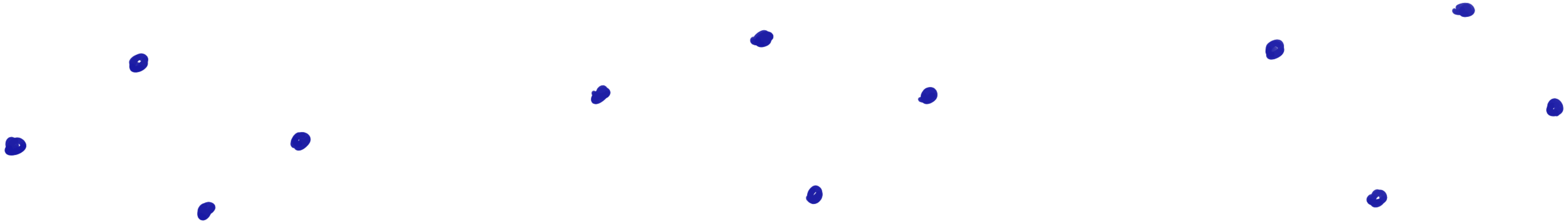

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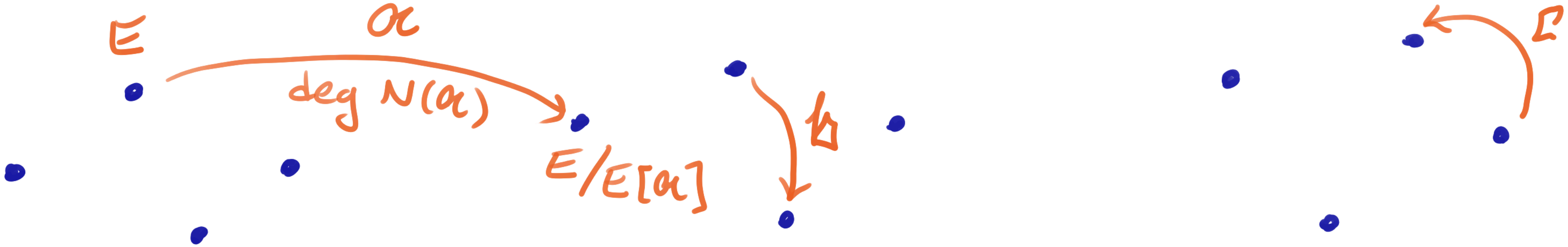

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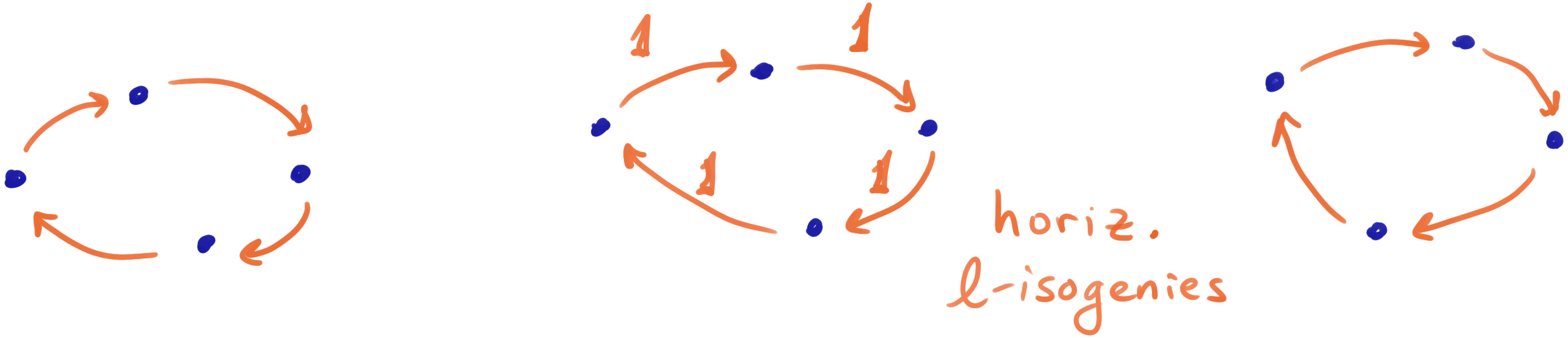

$(\ell) = \mathfrak{l}\cdot\bar{\mathfrak{l}}$

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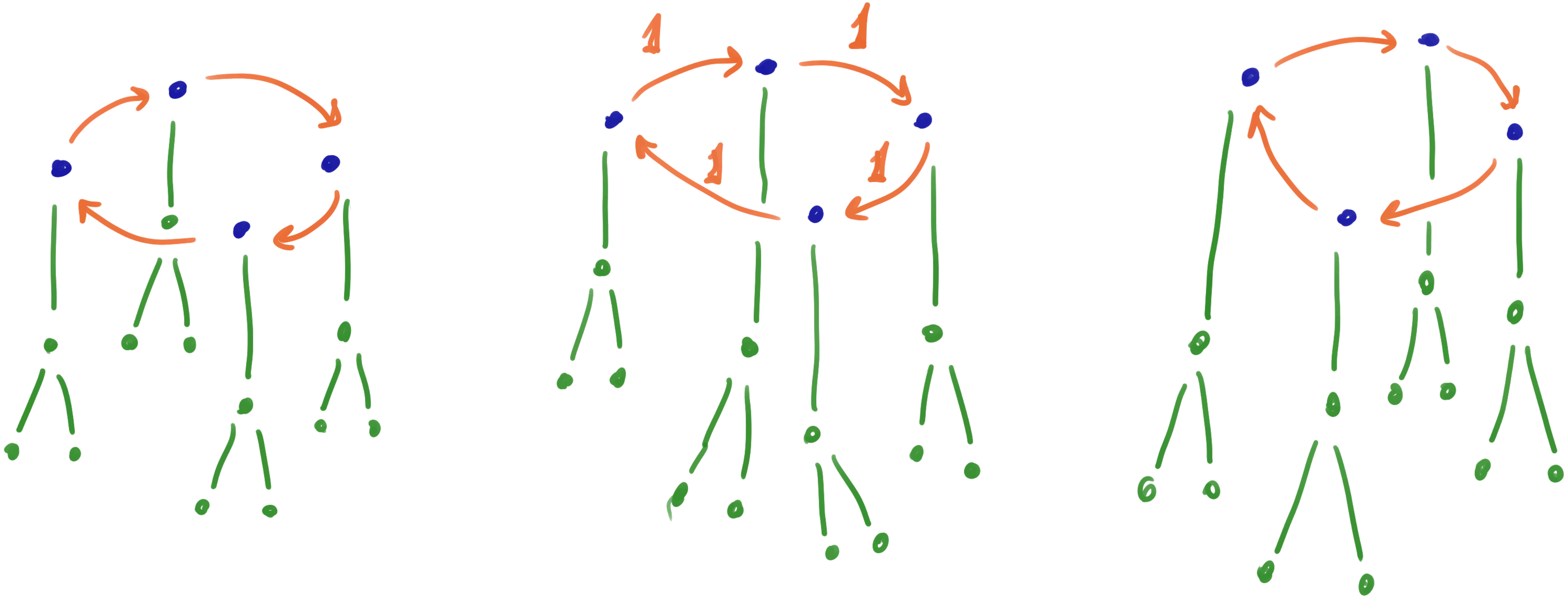

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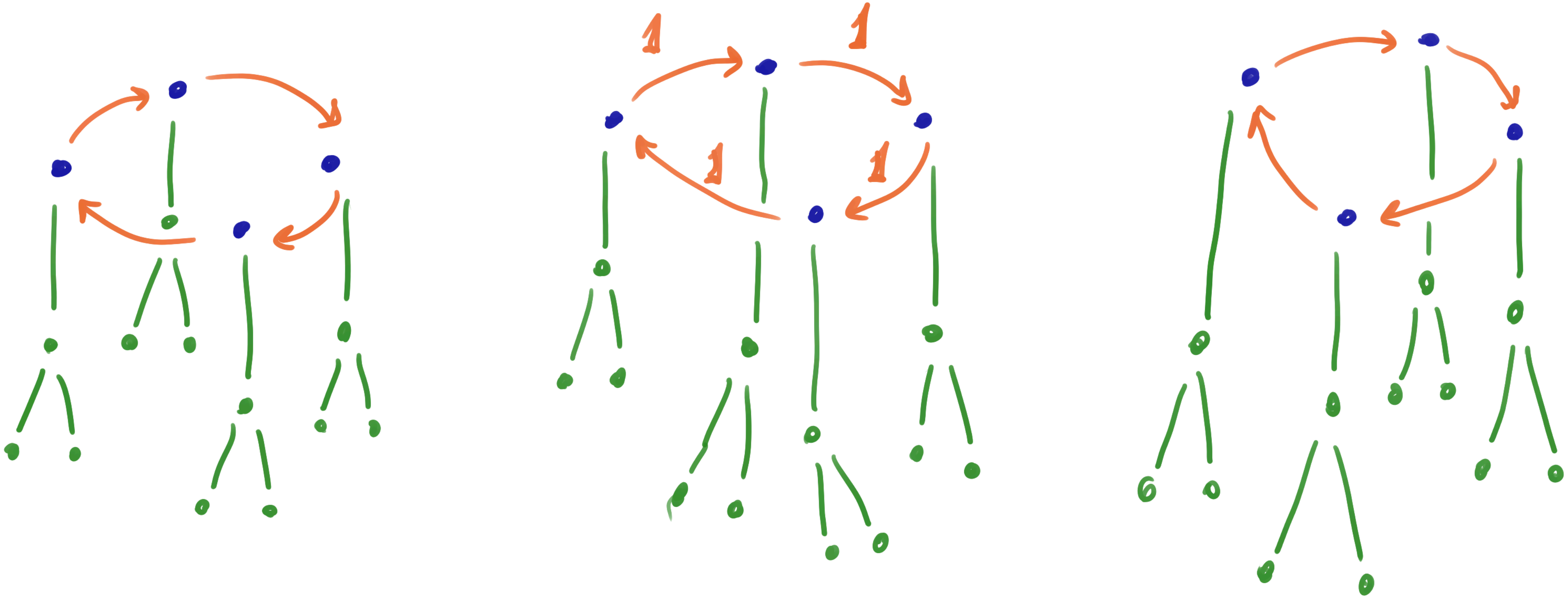
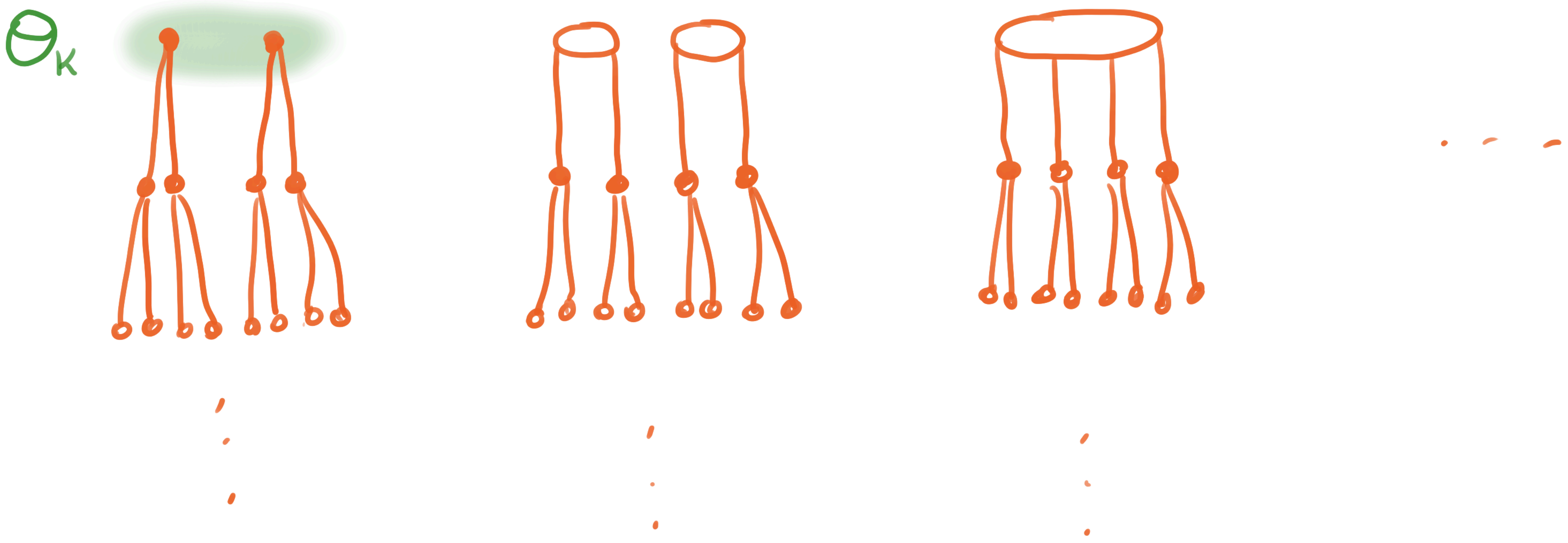


rim size = order of $[\mathbb{1}]$ in $Cl(\mathcal{O})$

K-oriented ss. isog. graph



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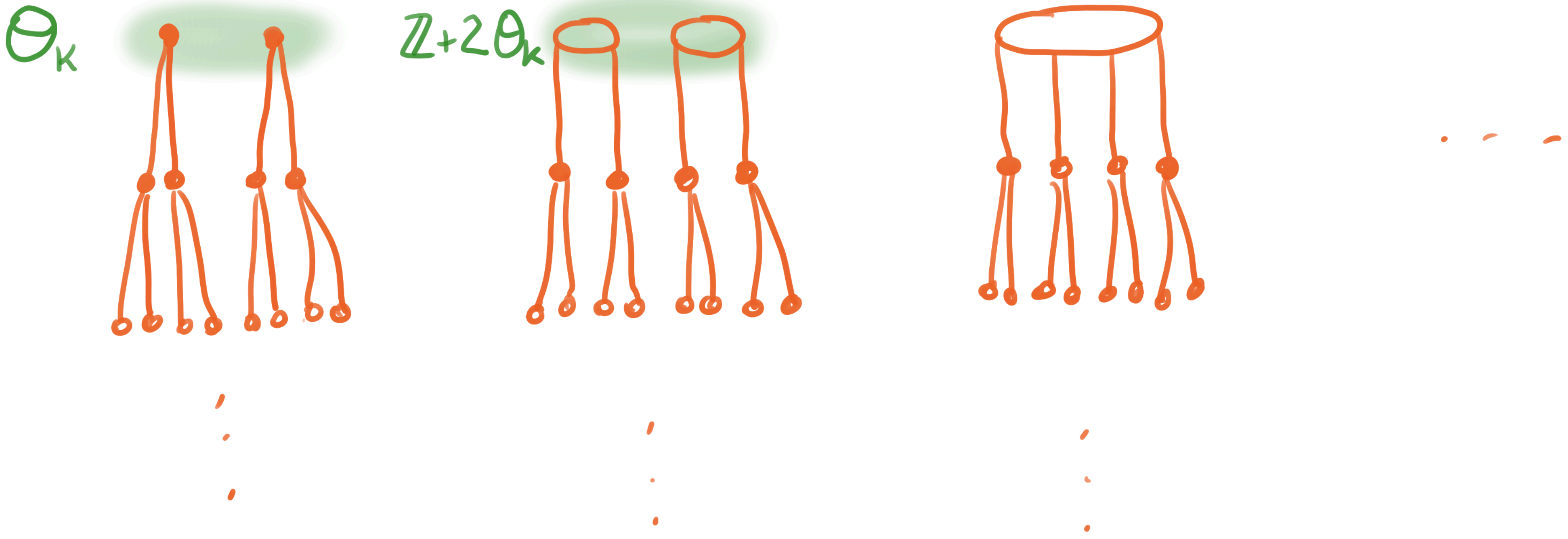

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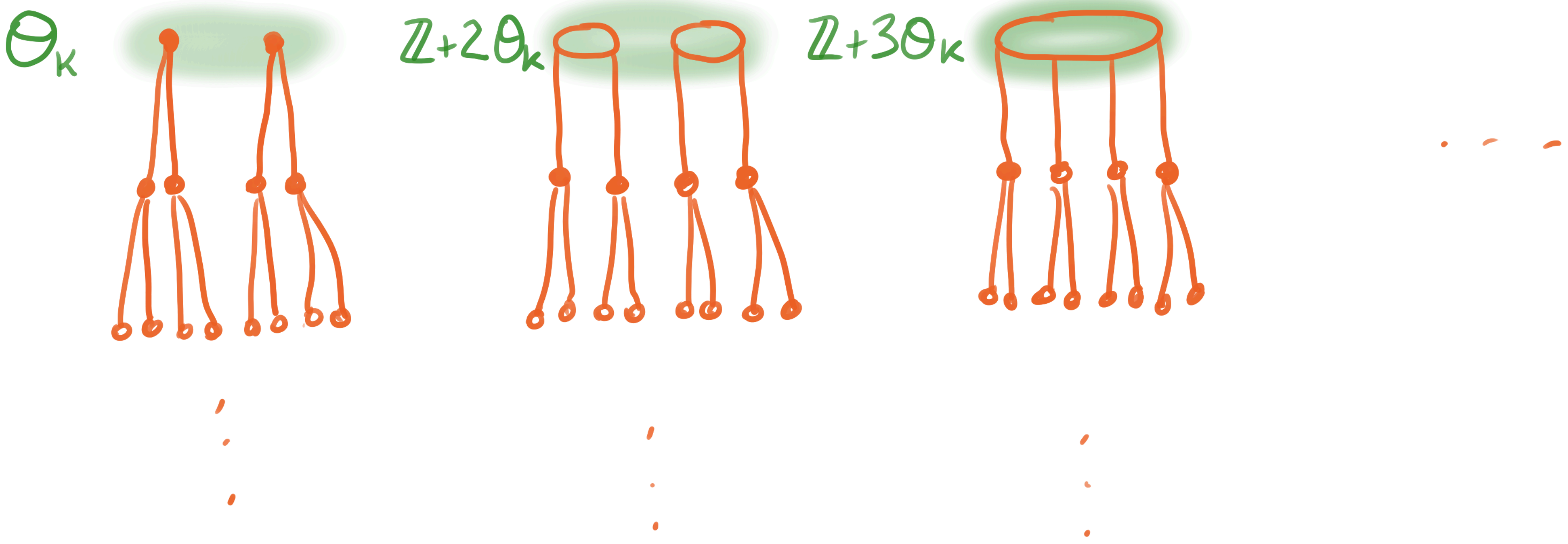

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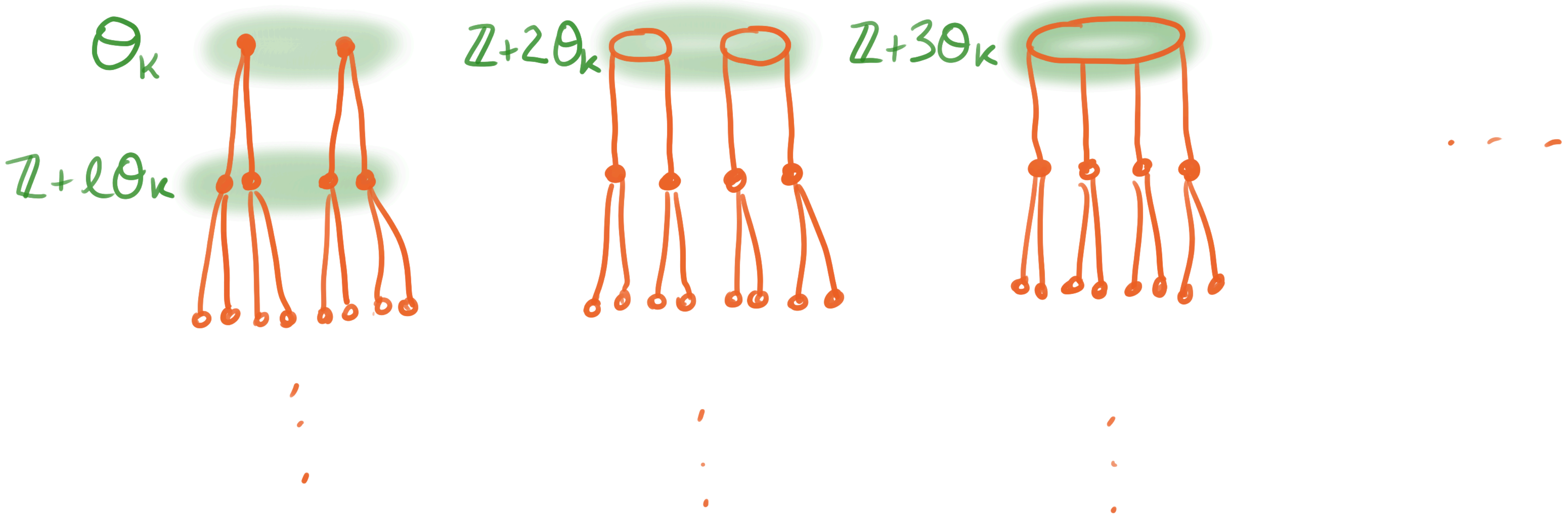

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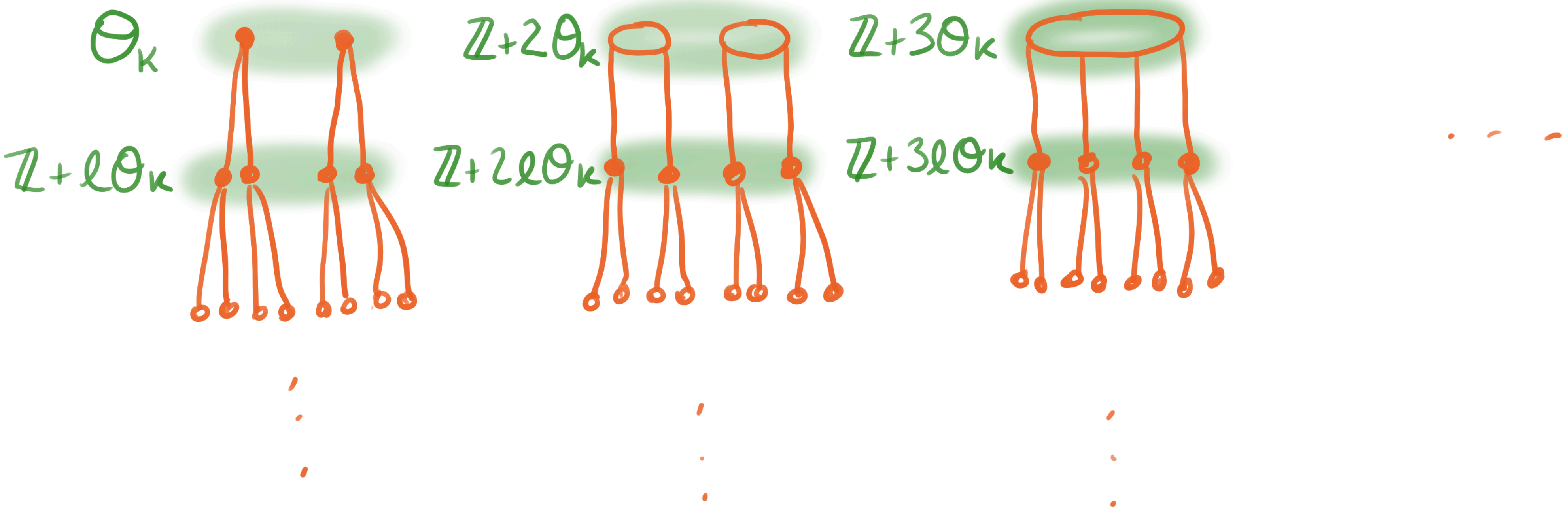

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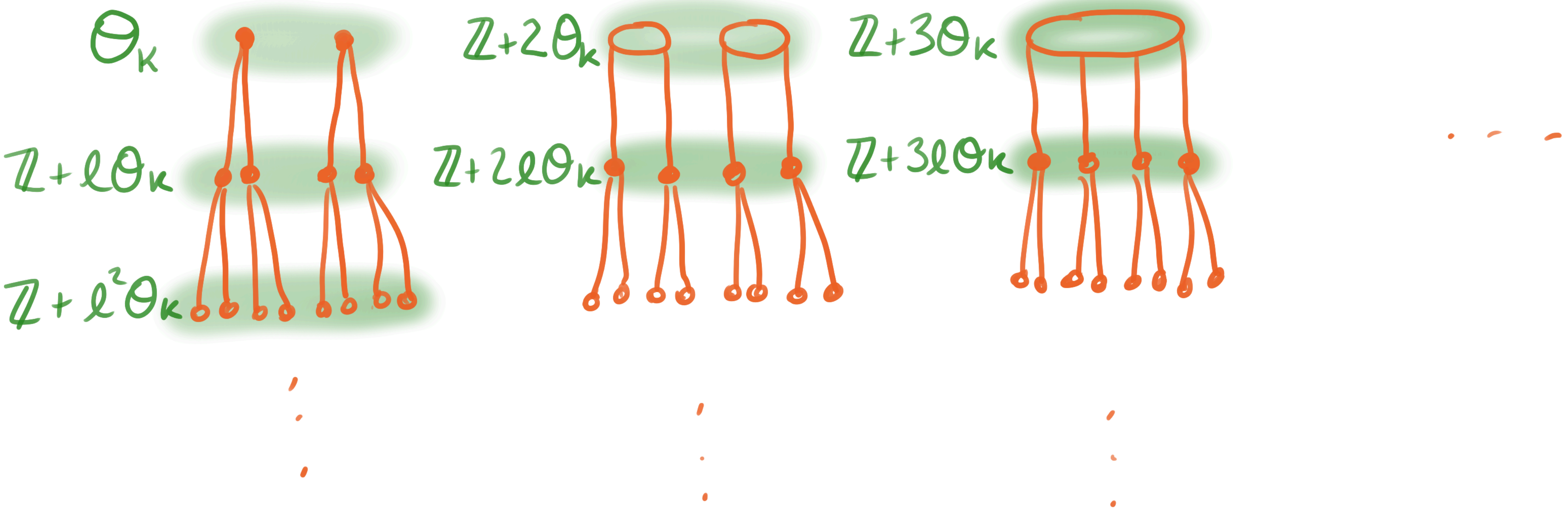
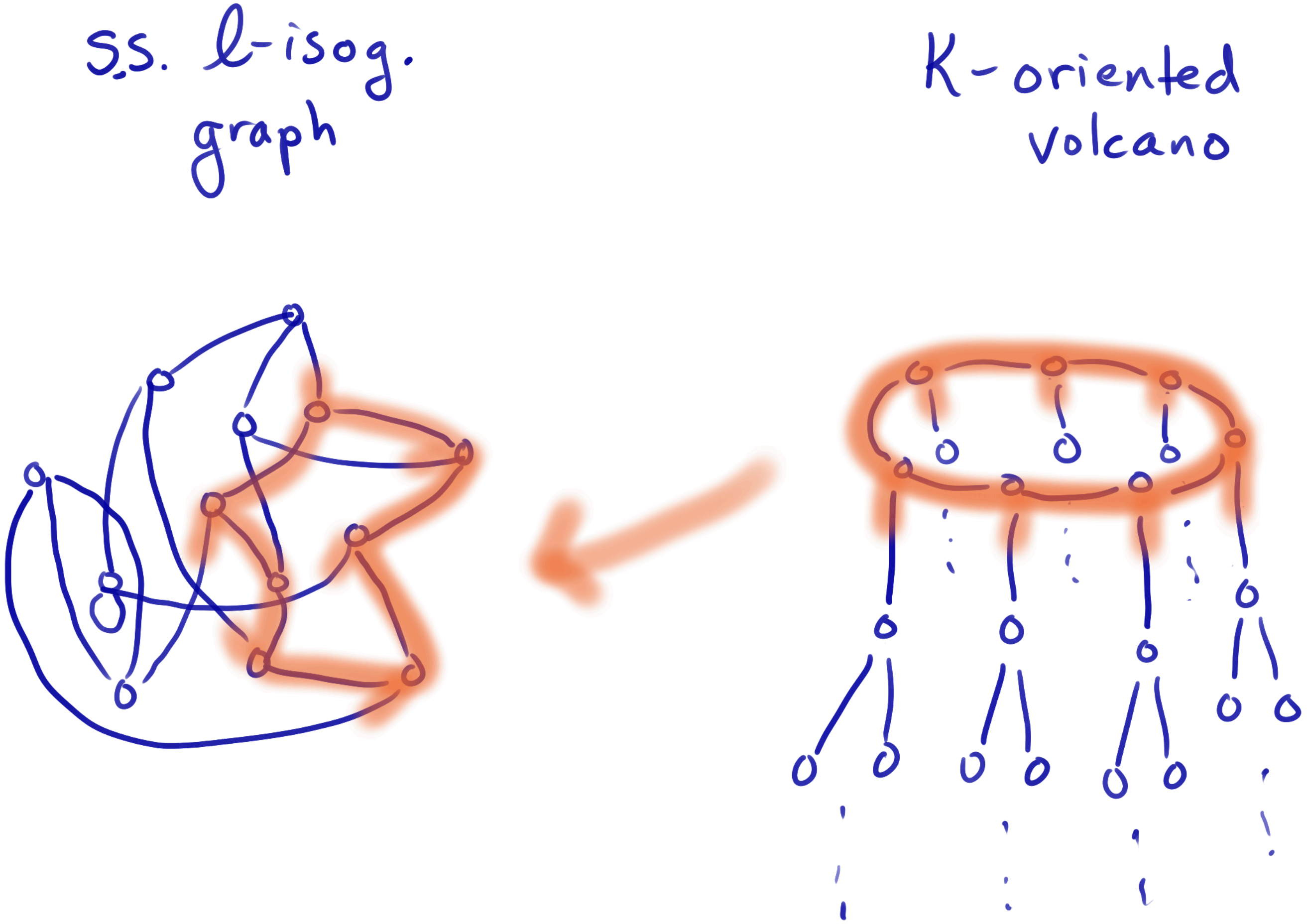


ascending → order gets bigger by index ℓ

descending → " " smaller " " "

horizontal → no change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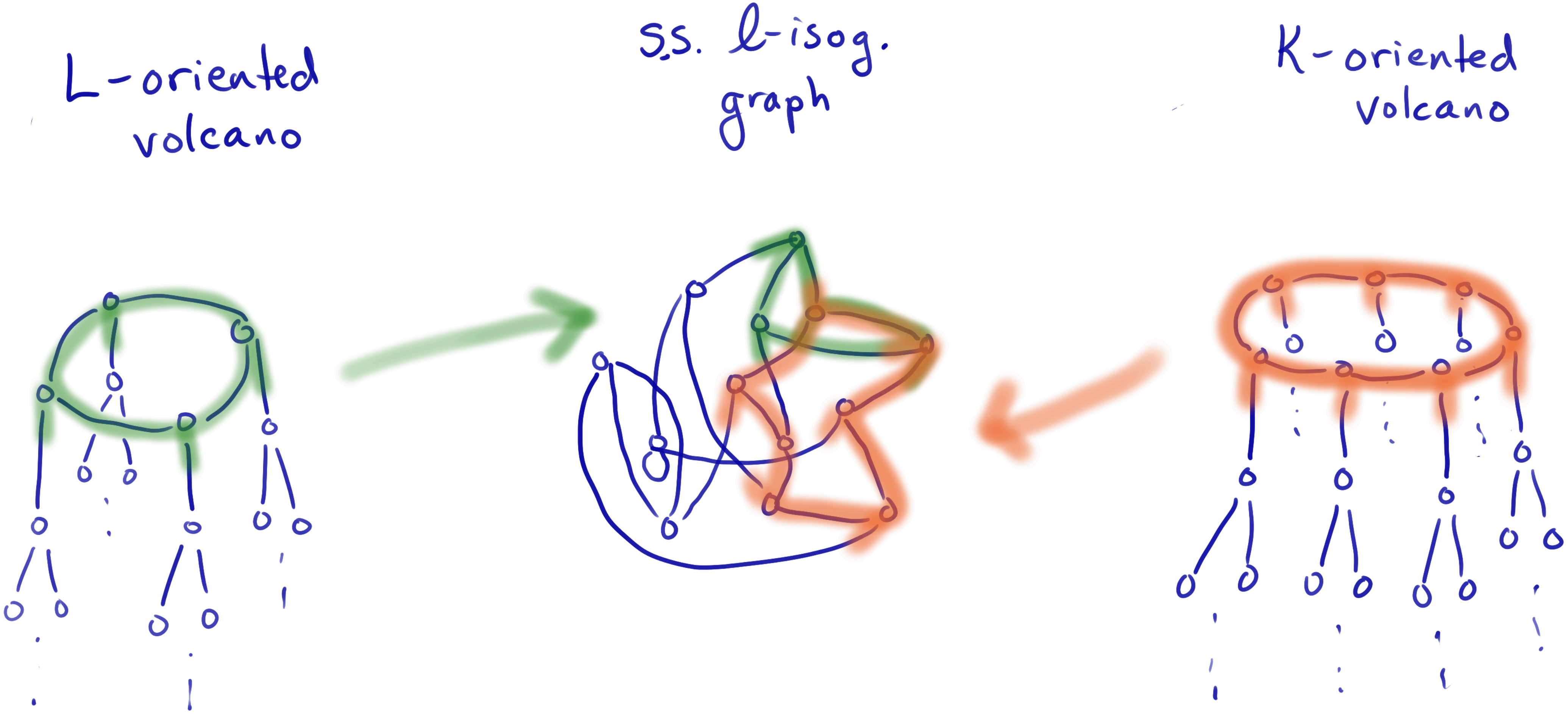
L-oriented volcano
s.s. ℓ-isog. graph
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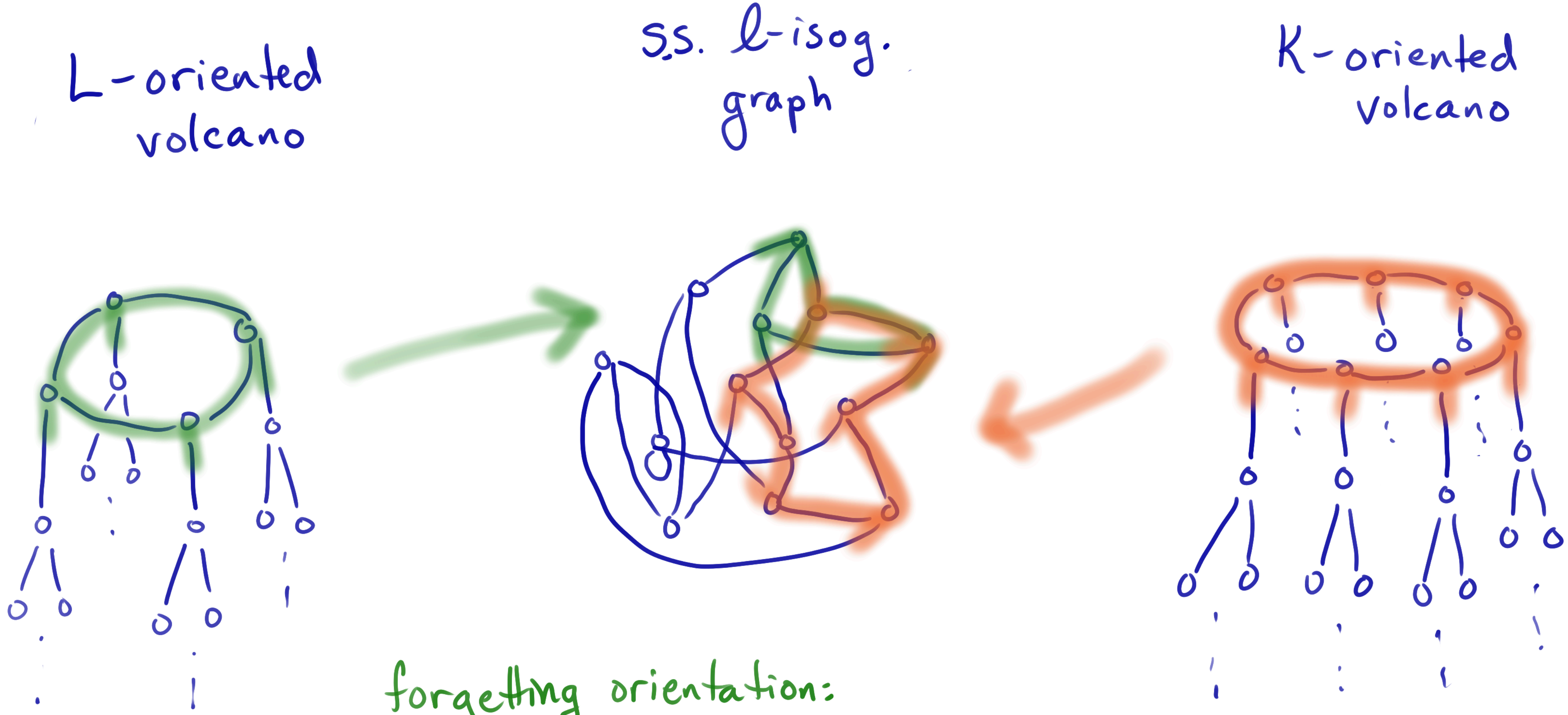
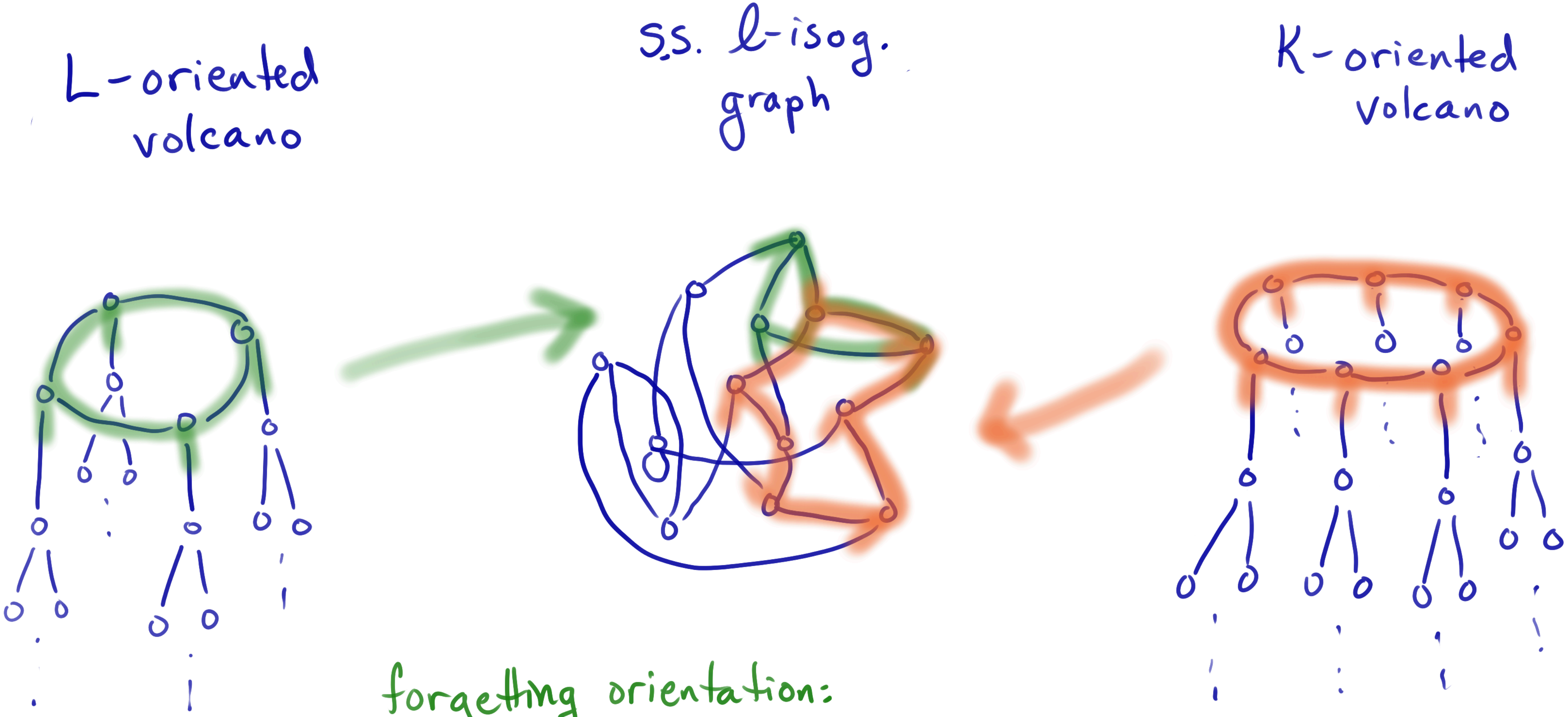


forgetting orientation:

rim → closed walk, same length

- no backtracking
- not a repeat of a smaller cycle



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"isogeny cycle"

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Primes $\ell < p$. Integer $r > 2$. $\mathcal{G}_\ell =$ s.s. ℓ-isog. graph

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up to conjugation

$\iota \longrightarrow \bar{\iota}$

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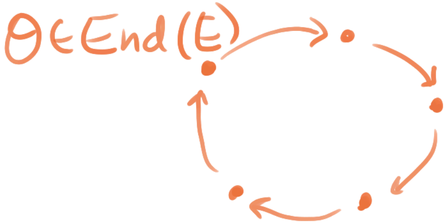

$\xrightarrow[\text{around cycle}]{\text{compose}}$

$\iota : \alpha \longmapsto \theta$

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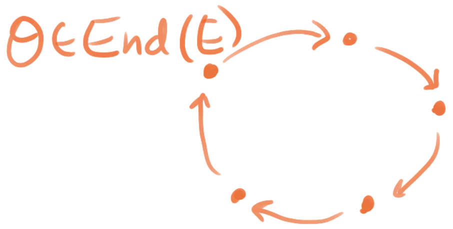

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θ :

- p not split
- $p \nmid$ conductor
- Δ_θ is ℓ-fund.
- $[\mathfrak{l}]$ has order r

Corollary: Counting Isogeny Cycles

1 or 2

class #

$$\frac{\ell^r}{2r} \overset{r\to\infty}{\sim} \#\text{ isogeny cycles of length } r = \frac{1}{r}\sum_{\theta} \varepsilon_{\theta,\ell}\, h_\theta \le \text{explicit bd}$$

Ramanujan graphs

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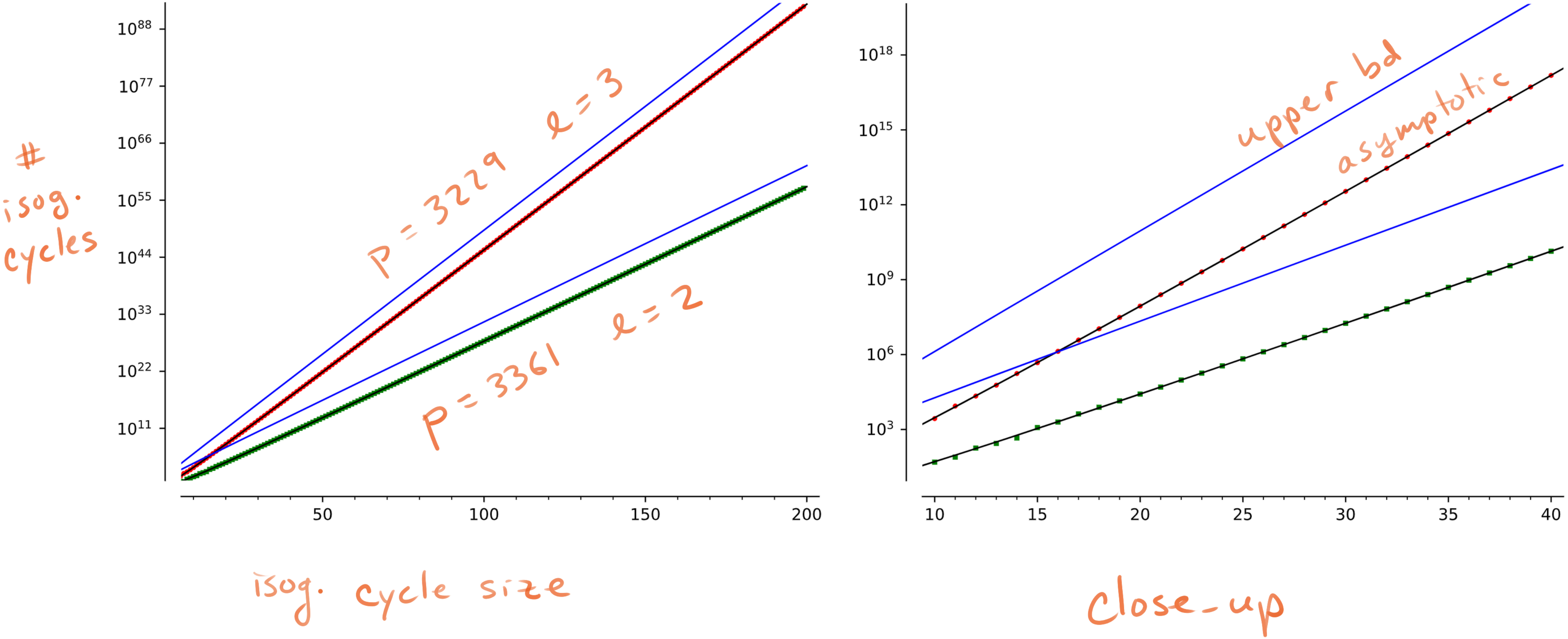
- p not split
- $p \nmid$ conductor
- Δ_θ is ℓ-fund.
- $[1]$ has order r

class # bds,

Möbius inversion

approx. $\ell^r \log^r$



isog. cycles
10^88
10^77
10^66
10^55
10^44
10^33
10^22
10^11
p = 3229 ℓ = 3
p = 3361 ℓ = 2
50
100
150
200
isog. cycle size
10^18
10^15
10^12
10^9
10^6
10^3
upper bd
asymptotic
10
15
20
25
30
35
40
close-up

Can you use this to navigate in the ℓ-isogeny graph?

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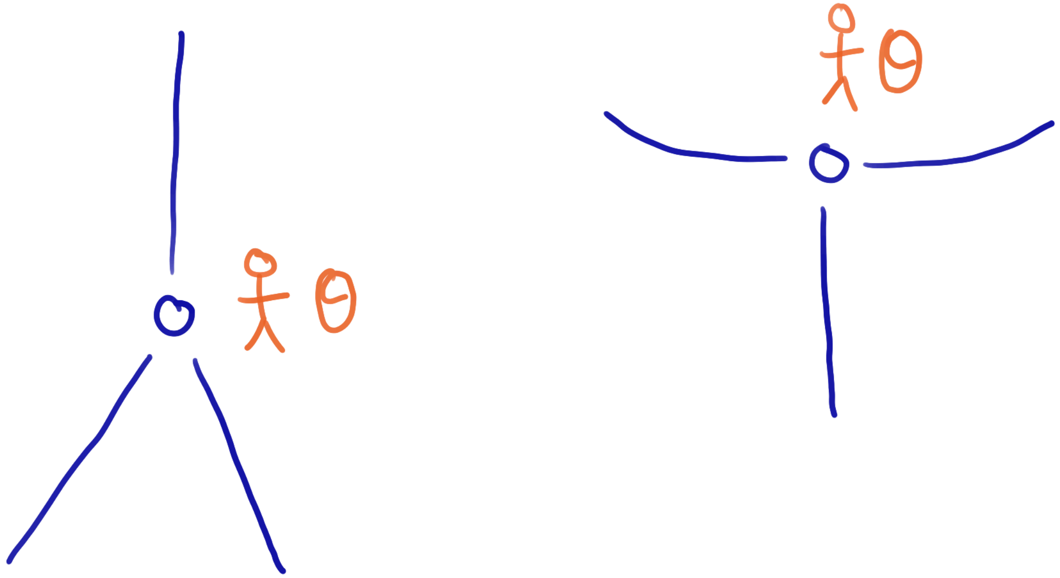

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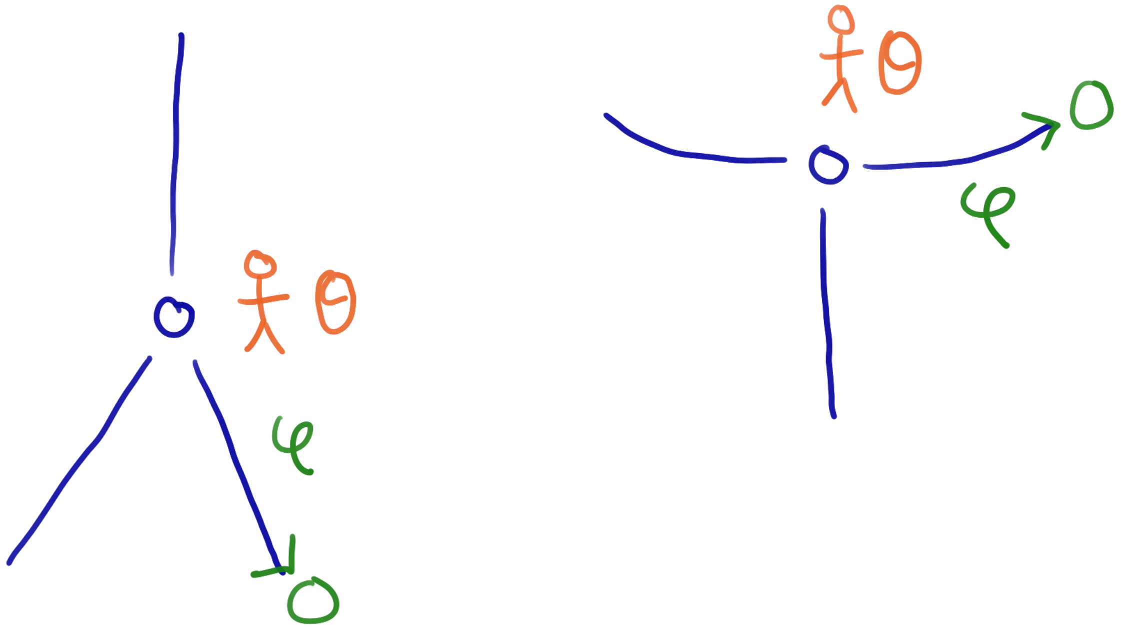

try one direction φ:

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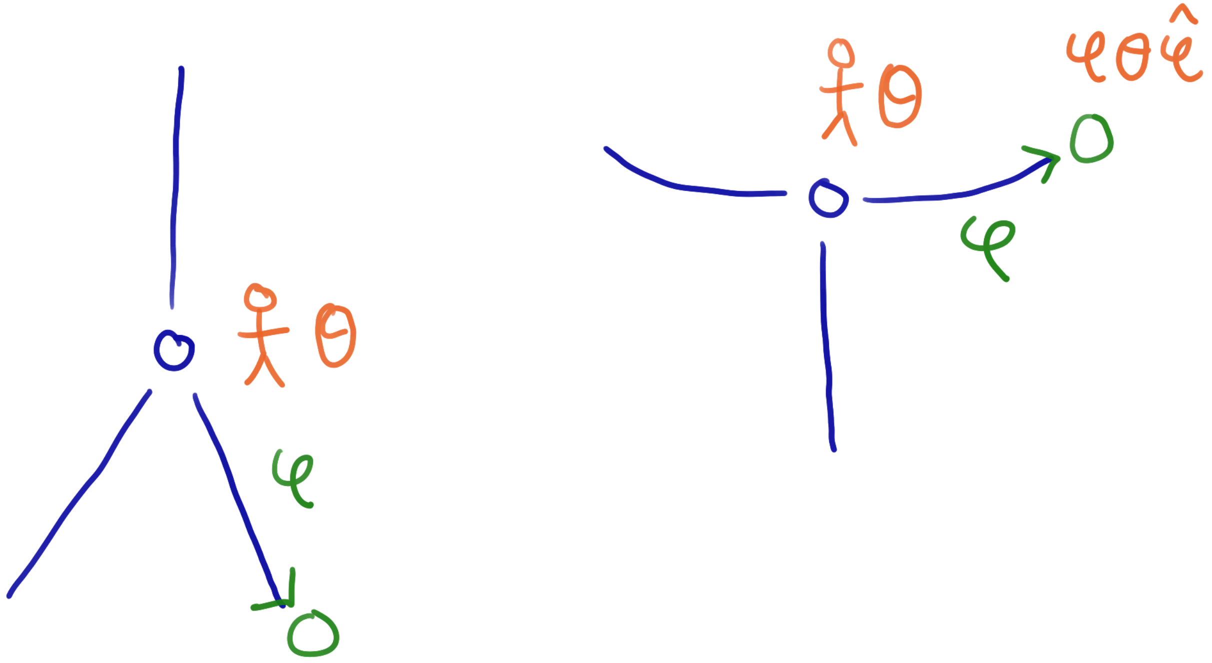

try one direction φ:

- the new orientation is

$$\varphi\,\theta\,\hat{\varphi}$$

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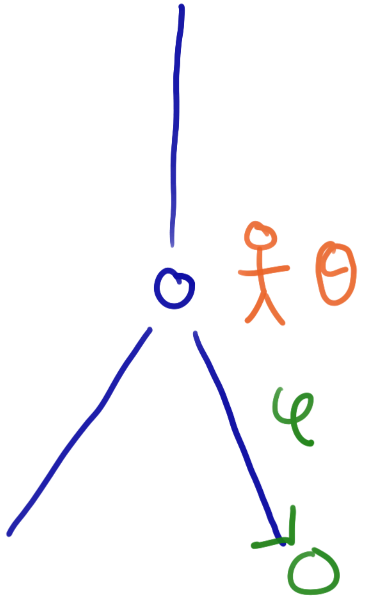

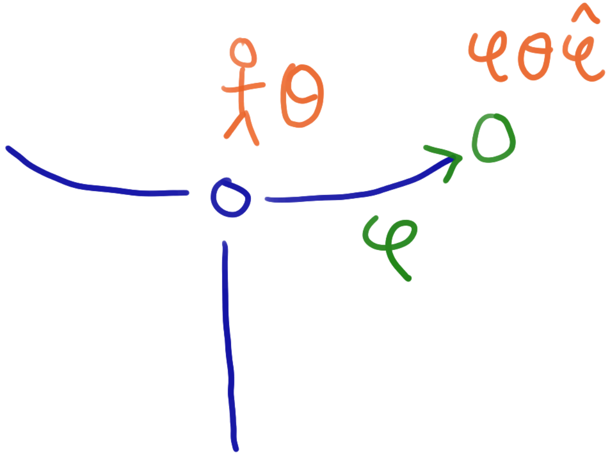

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- the new orientation is

$$\varphi\theta\hat{\varphi}$$

$\ell^2 \,\|\, \varphi\theta\hat{\varphi} \Leftrightarrow$ ascending

$\ell \,\|\, \varphi\theta\hat{\varphi} \Leftrightarrow$ horizontal

$\ell \nmid \varphi\theta\hat{\varphi} \Leftrightarrow$ descending

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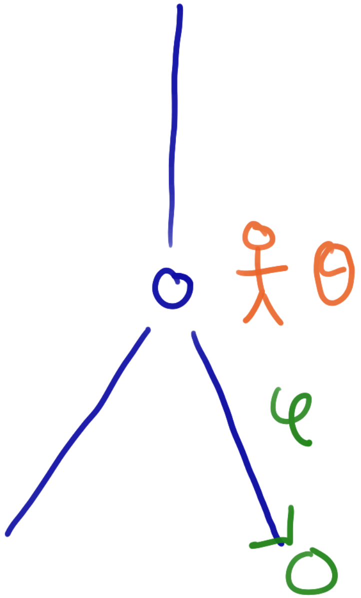

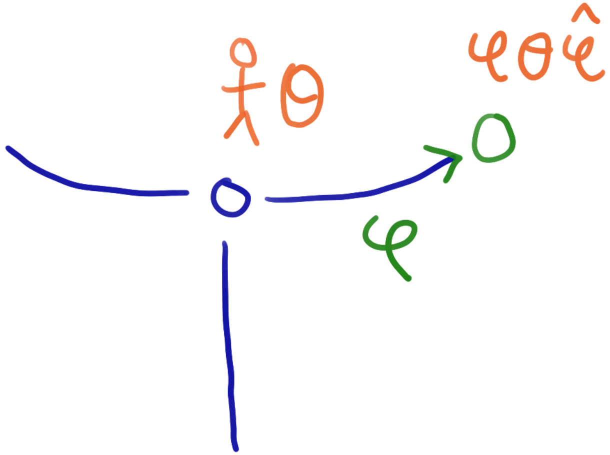

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* This works if θ is ℓ-primitive & ℓ-suitable

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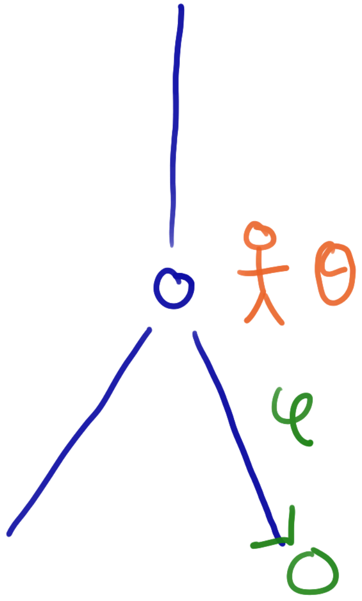

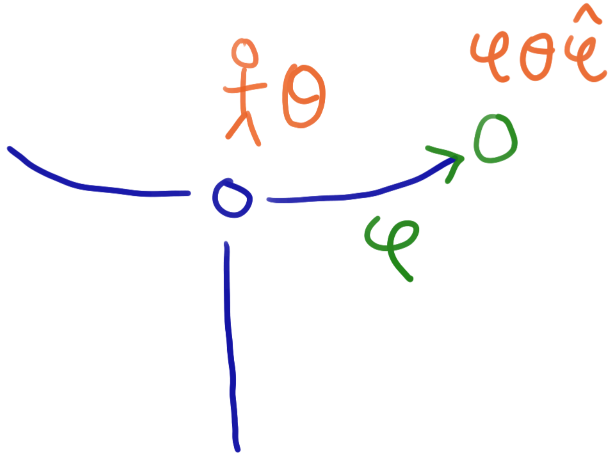

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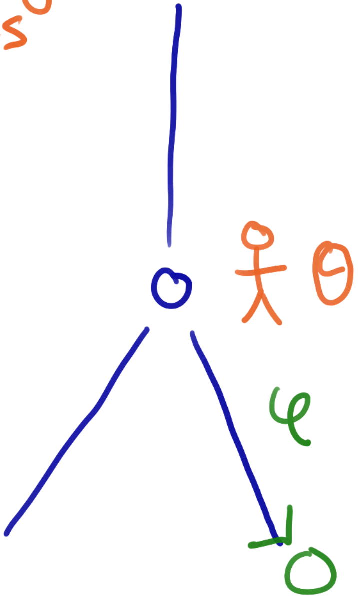
☆ This works if θ is ℓ-primitive & ℓ-suitable

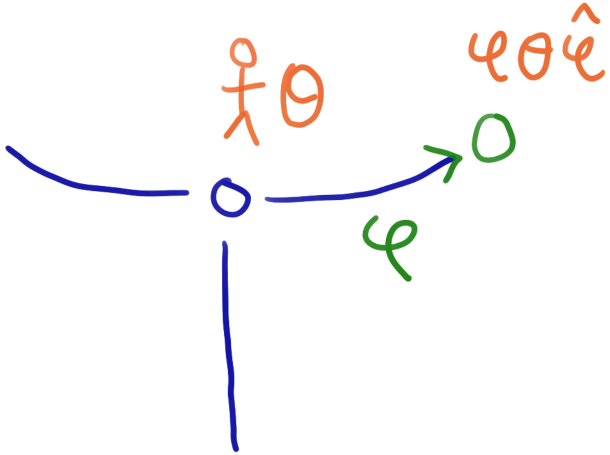
not div. by ℓ

fix by translation $\theta \mapsto \theta + n$

Can you use this to navigate in the ℓ-isogeny graph?

∃ another method using eigenvalues of θ





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⚠ This works if θ is <u>ℓ-primitive</u> & <u>ℓ-suitable</u>

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fix by translation $\theta \mapsto \theta + n$

(ACLSST)

Thm There exists a classical algorithm:

Input: $\underset{\in \mathrm{End}(E)}{\Theta}$ with $|\Delta'| \leq p^2$, can be efficiently evaluated on pts.

Output: ℓ-isog. path of length $O(\log p + h_{\Delta'})$ from E to $j=1728$.

Runtime: $h_{\Delta'} L_d(\frac{1}{2})\,\mathrm{poly}(\log p)$

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ℓ-fundamental part of Δ_θ

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class #

$e^{O((\log d)^{1/2} (\log\log d)^{1/2})}$

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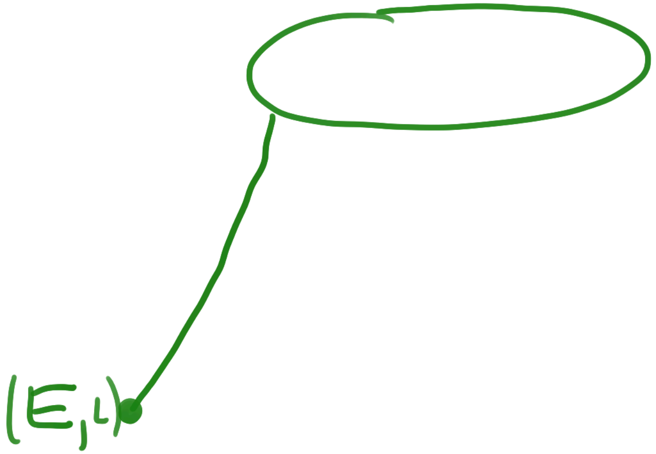

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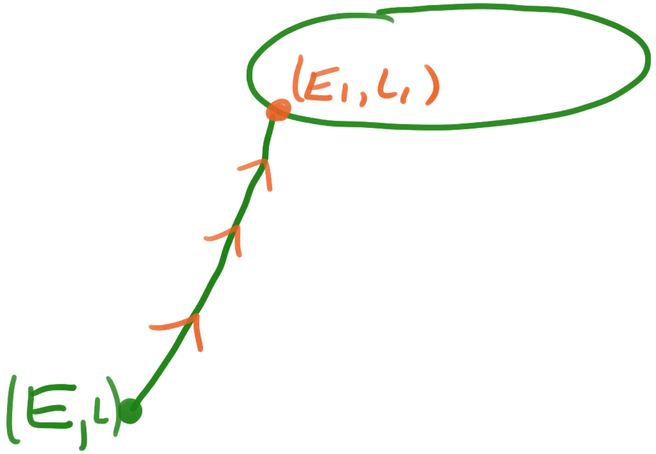

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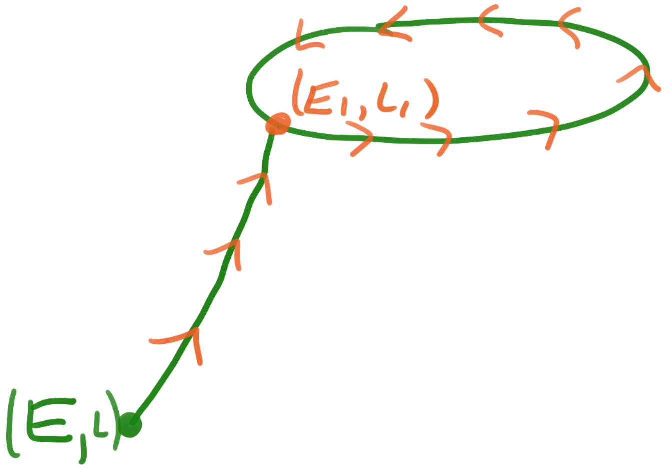

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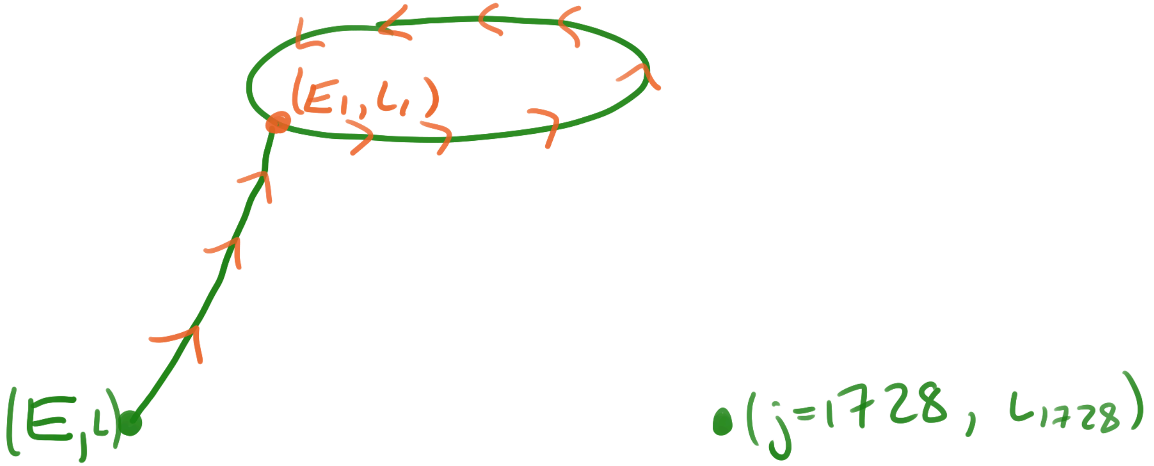

} <u>Subalgorithm:</u> orient 1728 by a given field

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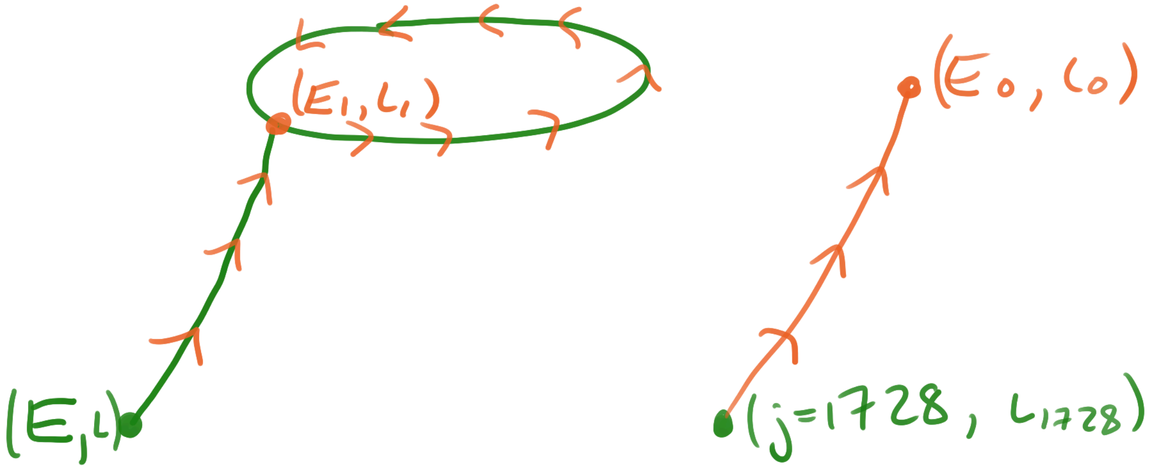

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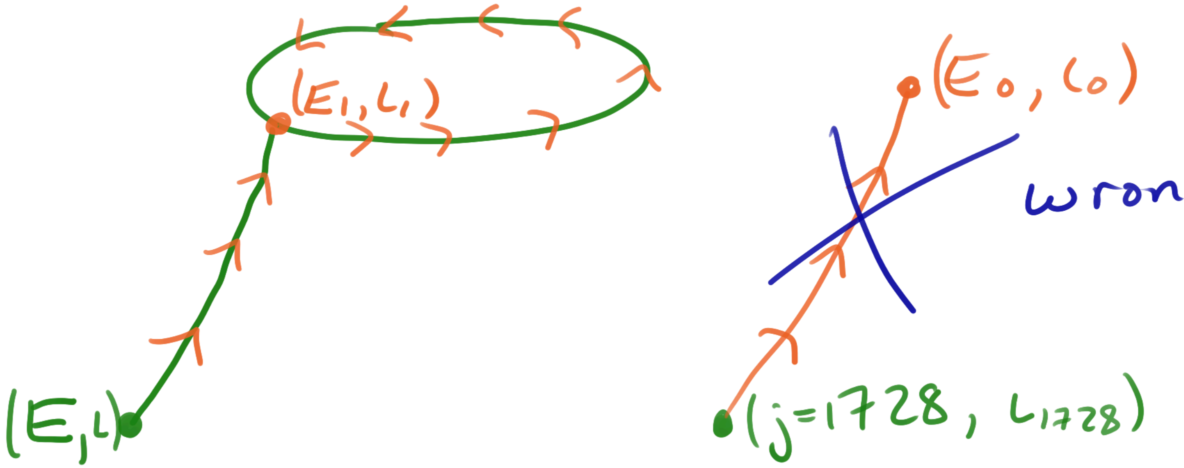

wrong volcano

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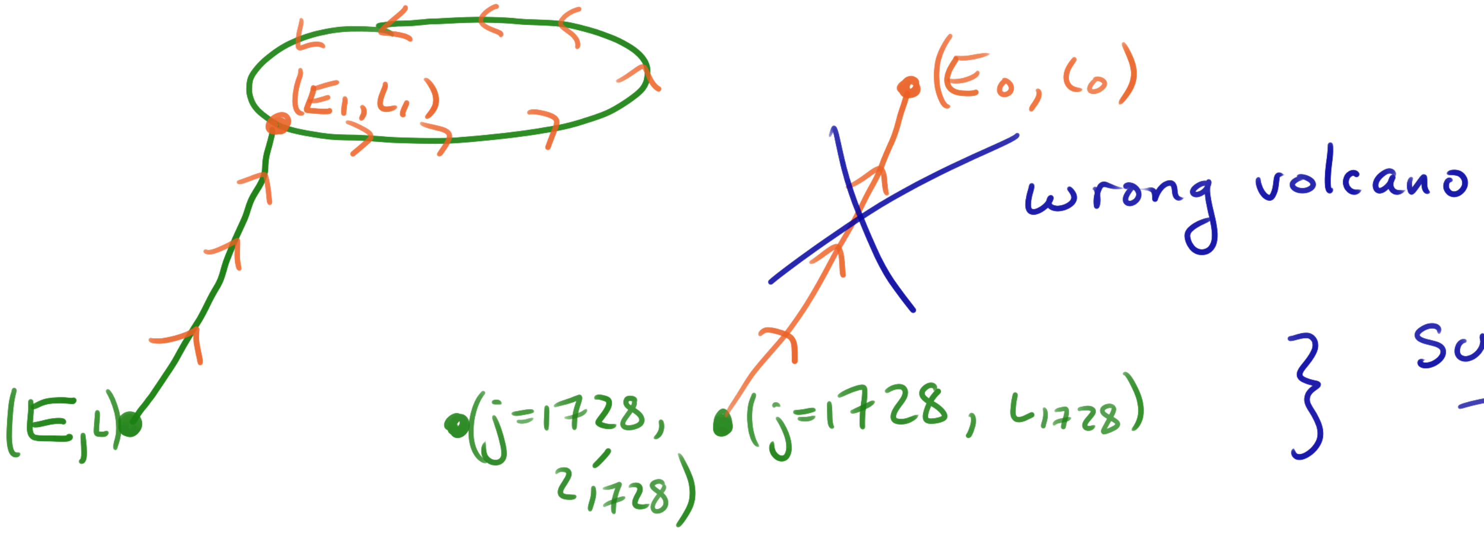

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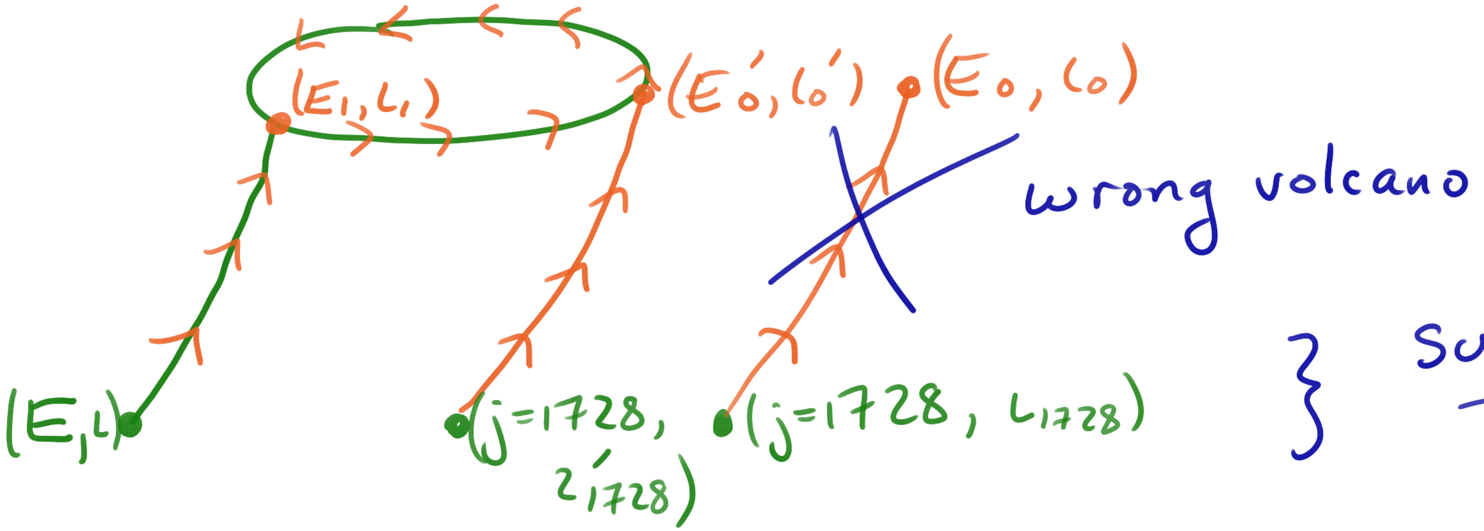

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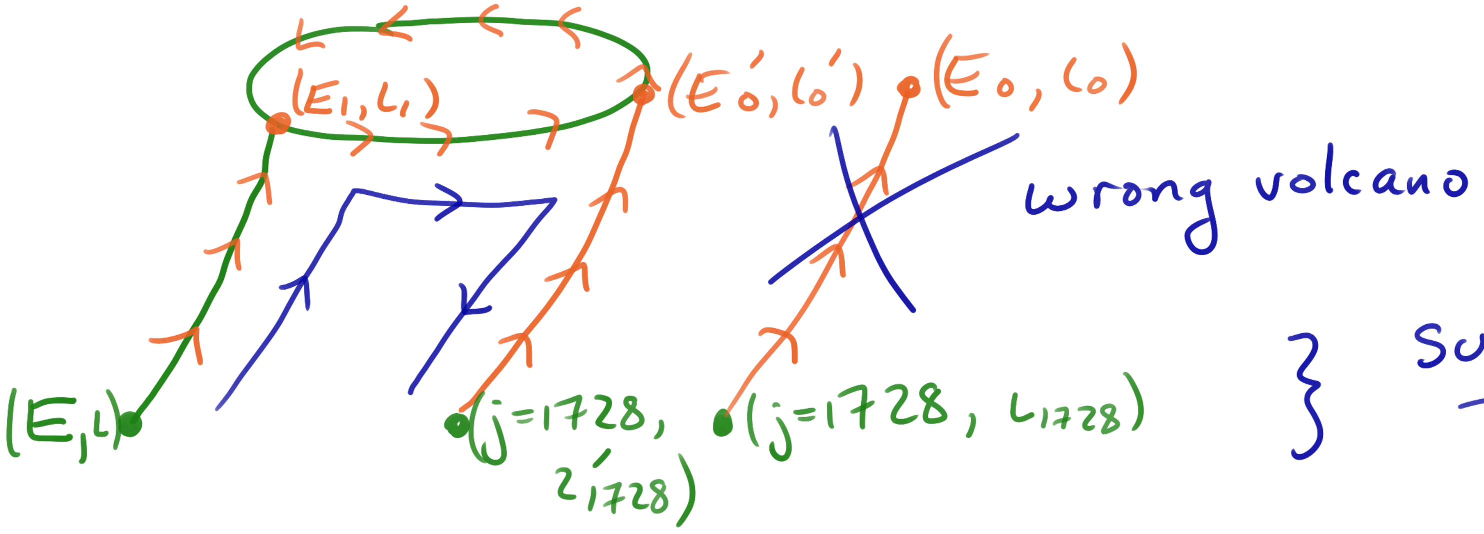

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walking the rim

(expected size $h_{\Delta'}$)

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at each step, need to translate

$$\theta \mapsto \theta + n$$

that is
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- powersmooth degree

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solution: sieving (subexp.)

(ACLSST)

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Input: $\theta \in \mathrm{End}(E)$ with $d \ll |\Delta| \leq p^2$, efficient to evaluate
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Runtime: $L_{|\Delta|}(\frac{1}{2})\,\mathrm{poly}(\log p)$

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1) Find Δ' s.t. (E,θ) is $\mathcal{O}_{\Delta'}$-primitive (quantum HSP)

2) Repeat:
- orient 1728
- walk to rim (E_1, ι_1)
- check rim is $\mathcal{O}_{\Delta'}$-primitive (quantum HSP)

3) Find $[\mathfrak{a}] \in \mathcal{Cl}(\mathcal{O}_{\Delta'})$ s.t. $[\mathfrak{a}] \cdot (E, \iota_\theta) = (E_1, \iota_1)$

(quantum, adaptation of Childs-Jao-Soukharev)

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