Visualizing imaginary quadratic fields

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Imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, for integers $d > 0$, are perhaps the simplest number fields after $\mathbb{Q}$. They are equal parts helpful first example and misleading special case. Like $\mathbb{Z}$, the Gaussian integers $\mathbb{Z}[i]$ (the case $d = 1$) have unique factorization and a Euclidean algorithm. As d grows, however, these properties eventually fail, first the latter and then the former.

The classical Euclidean algorithm (in $\mathbb{Z}$) expresses any element of $\mathrm{SL}_2(\mathbb{Z})$ as a product of elementary matrices in $\mathrm{SL}_2(\mathbb{Z})$. It is remarkable that among number fields K (whose rings of integers we denote $\mathcal{O}_K$), $\mathrm{SL}_2(\mathcal{O}_K)$ fails to be generated by elementary matrices exactly when K is a non-Euclidean imaginary quadratic field [1, 10].

A particularly useful way to visualize the group $\mathrm{SL}_2(\mathbb{Z})$, or $\mathrm{PSL}_2(\mathbb{Z})$, is to study its action as Möbius transformations on the upper half plane, as in Figure 1. To study the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$, when K is imaginary quadratic, consider instead the upper half space H^3 lying above the complex plane. This is a model of hyperbolic space with boundary $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The hyperbolic isometries of this model are exactly the Möbius transformations, extended from $\widehat{\mathbb{C}}$. Each Bianchi group forms a discrete subgroup of hyperbolic isometries; in other words it is a Kleinian group. In analogy to Figure 1, each has a 3-dimensional fundamental region.

For today, however, let us focus on the boundary: consider the orbit of $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subseteq \widehat{\mathbb{C}}$. Möbius transformations take circles (including $\widehat{\mathbb{R}}$, a circle through ∞) to other circles. The full orbit of $\widehat{\mathbb{R}}$ is dense in the plane, but if we restrict ourselves to drawing only those circles having bounded curvature (recall that curvature is the reciprocal of radius), we obtain intricate images such as in Figure 2.

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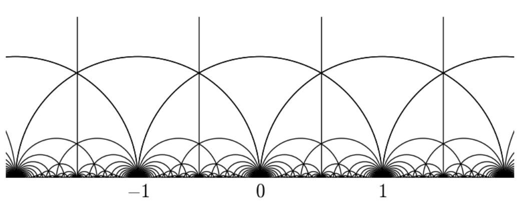

Figure 1. The upper half plane, tiled by images of a fundamental region for $\mathrm{PSL}_2(\mathbb{Z})$.

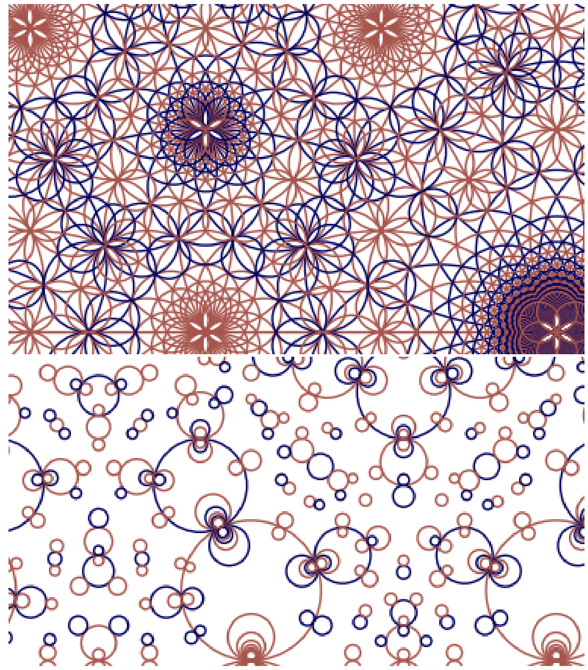

Figure 2. Schmidt arrangements of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-19})$. Both fields are class number one; only the former is Euclidean. Colour indicates the parity of the curvature.

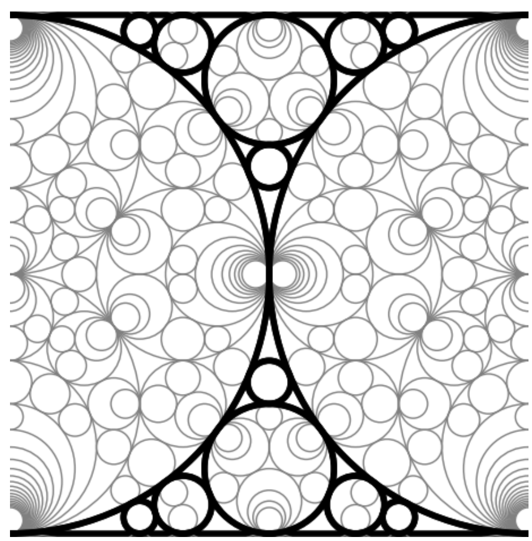

Figure 3. The unit square of $\mathcal{S}_{\mathbb{Q}(i)}$, showing curvatures ≤ 20, with the Apollonian strip packing in bold. In each pencil of circles tangent at a point, a circle has an 'immediate neighbour,' being the closest circle in the pencil, with disjoint interior. Apollonian circle packings are obtained by taking the closure of any one circle under such 'immediate tangency' [8].

We will call this the Schmidt arrangement $\mathcal{S}_K$ of K, for Asmus Schmidt's work on complex continued fractions, in which this picture first appears [6]. Schmidt's viewpoint is that the recursive subdivision of $\widehat{\mathbb{C}}$ into circles and triangles shown in Figure 3 is the natural analogue of the Farey subdivision of the real line. To approximate a complex number with Gaussian Farey fractions, one describes its 'address' in the Schmidt arrangement; nearby tangency points are good approximations.

Schmidt arrangments have a number of nice properties. After an appropriate scaling, all curvatures (inverse radii) are integral. Circles intersect only tangently, in all cases except the Eisenstein integers (where extra roots of unity add complication). At each K-rational point in $\widehat{\mathbb{C}}$, there is a pencil of circles whose curvatures form an arithmetic progression whose common difference is the norm of the denominator of the point. See [7].

The geometry of the Schmidt arrangement is controlled by the arithmetic of the field. For example, one can 'move' from circle to tangent circle by the use of elementary matrices in the Bianchi group, so one can see the Euclidean algorithm:

Theorem 0.1 (S. [7, Theorem 1.5])**.** $\mathcal{O}_K$ is Euclidean if and only if $\mathcal{S}_K$ is connected.

The circles themselves represent certain ideal classes.

Theorem 0.2 (S. [7, Theorem 1.4])**.** Let $f \geq 1$ be an integer, and let $\mathcal{O}_f$ be the order of conductor f in K. Circles of curvature $f\sqrt{-\Delta}$ in $\mathcal{S}_K$ (where $3 \neq \Delta < 0$ is the discriminant of K), up to translation by $\mathcal{O}_K$ and rotation by 180 degrees, are in bijection with the kernel of the natural map of ideal class groups $\operatorname{Pic}(\mathcal{O}_f) \to \operatorname{Pic}(\mathcal{O}_K)$.

In fact, a circle of curvature f obtained by applying

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K)$$

to $\widehat{\mathbb{R}}$ corresponds to the ideal class of $\mathcal{O}_f$ generated by the lattice $\beta\mathbb{Z} + \delta\mathbb{Z}$, which has covolume f.

The author's interest in Schmidt arrangements arose from the study of Apollonian circle packings; see Figure 4. There are many examples of Apollonian circle packings whose circles have only integer curvatures; it is conjectured that, except for certain congruence conditions, all sufficiently large integers appear in any such packing [3, 4]. The Apollonian group, which controls the curvatures, is a thin subgroup of $O_{3,1}$, and it represents the principle test case for new methods in thin groups. For an excellent overview, see [2].

The connection is that $\mathcal{S}_{\mathbb{Q}(i)}$ appeared, independently, in work of Graham, Lagarias, Mallows, Wilks and Yan as an Apollonian super-packing [5]. In general, it is possible to isolate an

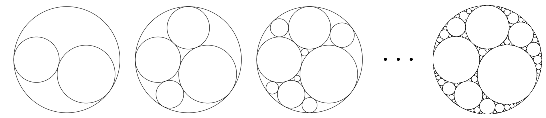

Figure 4. The iteration process generating an Apollonian circle packing from three mutually tangent circles.

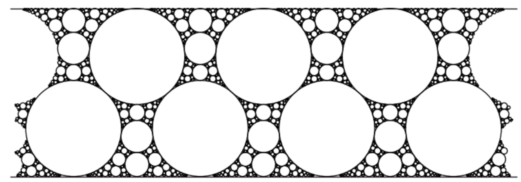

Figure 5. The $\mathbb{Q}(\sqrt{-7})$-Apollonian packing.

Apollonian-like circle packing – and therefore a thin subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of arithmetic interest – using the simple geometric criterion of Figure 3. See Figure 5, and [8].

A note on figures. The figures in this document were produced using Sage Mathematics Software [9].

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