

Legendre/Jacobi Cheat Sheet

Legendre Symbol

p odd prime, $a \not\equiv 0 \pmod{p}$

$$\left(\frac{a}{p}\right) = \begin{cases} +1 & \text{if } x^2 \equiv a \pmod{p} \text{ is solvable.} \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ is not solvable.} \end{cases}$$

Jacobi Symbol

n odd positive, a nonzero, $\gcd(n, a) = 1$, $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$

$$\left(\frac{a}{n}\right) = \underbrace{\left(\frac{a}{p_1}\right)^{e_1} \left(\frac{a}{p_2}\right)^{e_2} \dots \left(\frac{a}{p_k}\right)^{e_k}}_{\text{Legendre symbols}}$$

Properties (Assuming all symbols are defined)

① If $a \equiv b \pmod{n}$, then $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

② If p is prime (Legendre symbol only), then $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

③ $\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right) \left(\frac{b}{n}\right)$

④ $\left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}$

⑤ $\left(\frac{2}{n}\right) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \pmod{8} \\ -1 & \text{if } n \equiv \pm 3 \pmod{8} \end{cases}$

⑥ "Quadratic Reciprocity"

$$\left(\frac{m}{n}\right) = \begin{cases} -\left(\frac{n}{m}\right) & \text{if } m \equiv n \equiv 3 \pmod{4} \\ +\left(\frac{n}{m}\right) & \text{otherwise} \end{cases}$$