Mathematics 3130
Show H is a subspace of V
What it looks like in my head

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This document is intended to serve as a sort of diagram of my thought process when I am showing something is a subspace of a vector space. It is meant as an example of the mental work that goes into a logical argument. The final argument, written up, may in fact be very short by comparison to all the steps and pieces shown here. But you do not truly understand the solution unless you probe every aspect of it in detail. This is that detail. It serves as an example of the sort of questions you can invent for yourself to work through a tough problem.

Note: Another type of proof will have similar but different steps in its 'mental diagram'. This is an example, **not a recipe**. It is meant to help you create your own 'diagrams' for your own purposes. In particular, you can probably create a proof that will get nearly full marks using this diagram without understanding what you're doing. **That's pointless.** I won't give you a diagram like this on exams and memorizing it is foolhardy. The point is to use the diagram as a guide to help fill out your own understanding. If I find that it serves as a plug-n-chug recipe for students, I will go home and cry myself to sleep.

1. **What is V?** Before we can address anything, we need to know about the world we are working in.
   (a) **What are its vectors?**
      i. **What type of objects are the vectors?** Example answers: functions from the real numbers to the real numbers, polynomials, 2×2 matrices, 3-coordinate vectors (the old fashioned 'vectors'), etc.

      ii. **What restrictions?** What properties must the objects have in order to be in V? Example answer: be continuous (of a function), be of degree at most 3 (of a polynomial), have last entry 0 (of a column vector), etc.

      iii. **Give three examples.** You don't understand something if you can't give examples of it.

iv. **Check they are in fact examples.** For each example, double check that it is of the type specified (see (1(a)i)) and satisfies the restrictions specified (see (1(a)ii)).

v. **Write V in set notation.** If it wasn't already given in this form, then write it in $V = \{\text{type} : \text{restrictions}\}$ notation. An example would be: $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$.

(b) **How do you add vectors?**

i. **Write down a notation for a general vector.** This means a letter or symbol, not an example. Example answers include: function $f = f(x)$, polynomial $p(t) = at^2 + b$, vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, etc.

ii. **Write down a notation for a different general vector.**

iii. **Write down their sum and its meaning.**

iv. **Give an example.** Choose specific vectors for the two general things, and compute their sum.

(c) **How do you scalar multiply vectors?**

i. **Write down a notation for a general vector.**

ii. **Write down a notation for a general scalar.** Why not always just use $c \in \mathbb{R}$. It's easy.

iii. **Write down their product and its meaning.**

iv. **Give an example.** Choose a specific vector and a specific scalar, and compute the product.

(d) **What is the zero vector?**

i. **Make a guess.** It's often the 'zero thing' whatever you think that might mean.

ii. **Is your guess a vector?** Compare to part (1a)

iii. **Does your guess satisfy the property $\mathbf{0} + \blacksquare = \blacksquare$?** That is, if you add your purported zero vector to any general vector, does it leave it unchanged? This is the defining property of the zero vector. So compute the sum of your zero vector and a general vector. You will need to use part (1(b)iii) to do the computation. Then see if the result is the same general vector.

iv. **Did you guess right?** If the guess you make in part (1(d)i) passes the two tests (1(d)ii), (1(d)iii), then you guessed right.

2. **What is H?**

(a) **What are its vectors?** This is just like (1a), but for H.

i. **What type of objects are the vectors?**

ii. **What restrictions?**

iii. **Give three examples.**

iv. **Check they are in fact examples.**

v. **How is this different from** (1a)**?** These questions were the same as in part (1a); where, if at all, do your answers differ? That's what makes H different from V.

3. **Is H a subset of V?** If not, it's not a subspace.

(a) **Are its vectors in V?** If the answer is yes, then it is a subset.

i. **Are they the same type of object as those in V?** Are (1(a)i) and (2(a)i) the same?

ii. **Do they satisfy the restrictions listed in** (1(a)ii)**?** Compare (1(a)ii) and (2(a)ii).

iii. **Are the examples in** (2(a)iii) **in V?**

iv. **What additional restrictions do elements of H satisfy?** That is, what is listed in (2(a)ii) that's not listed in (1(a)ii)?

v. **Write H in set notation as a subset of V.** If it wasn't already given in this form, then write it in $H = \{\mathbf{v} \in V : \text{additional restrictions}\}$ notation. An example would be: $H = \{f \in V : f(0) = 0\}$.

4. **Does H contain the zero vector?** If not, it's not a subspace.

   (a) **Remind me what the zero vector is.** See (1(d)i).

   (b) **Does $\mathbf{0}$ satisfy the restrictions of** (2(a)ii)**?** If so, then it is in H.

5. **Is H closed under addition?** If not, it's not a subspace.

   (a) **Write down a notation for a general vector.**

   (b) **Write down the extra criteria it satisfies because it is in H.** See (3(a)iv).

   (c) **Write down a notation for a different general vector.**

   (d) **Write down the extra criteria it satisfies because it is in H.** See (3(a)iv).

(e) **Write down their sum and its meaning.**

(f) **Check if the sum you just wrote down satisfies the restrictions of** (3(a)iv)**.** You should use the facts (5b) and (5d). If so, then it is in H. You've shown that two things in H, added, gives a result again in H. In other words, H is closed under addition.

6. **Is H closed under scalar multiplication?** If not, it's not a subspace.

   (a) **Write down a notation for a general vector.**

   (b) **Write down the extra criteria it satisfies because it is in H.** See (3(a)iv).

   (c) **Write down a notation for a general scalar.**

   (d) **Write down their product and its meaning.**

(e) **Check if the product you just wrote down satisfies the restrictions of** (3(a)iv)**.** You should use the fact (6b). If so, then it is in H. You've shown that a vector in H, scalar multiplied, gives a result again in H. In other words, H is closed under scalar multiplication.

7. **If H has passed all these tests, it is a subspace.** Why the heck do we show something is a subspace, again? Because subspaces are smaller vector spaces and when you're dealing with a subspace you can discuss dimension, basis, linear transformations, etc. If you are just dealing with a **subset**, not a **subspace**, then you can't talk about things like basis and you can't use your usual tools to solve problems. If you are mistaken, thinking it is a subspace when it is not, you may apply these tools, thinking you get a right answer, but end up with nonsense. **Now write up a proof.**