Minimal Strongly Abelian Varieties

Kearnes, Kiss, Szendrei

PALS 2020

Part of a larger project

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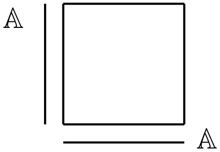

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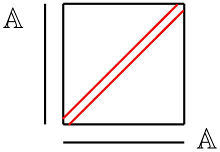

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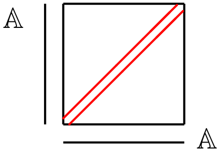

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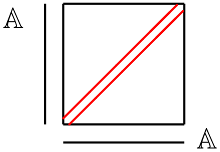

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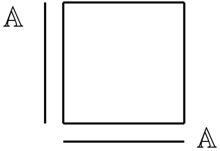

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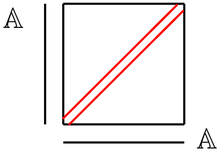

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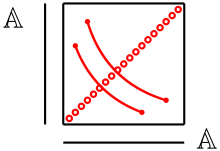

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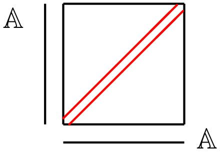

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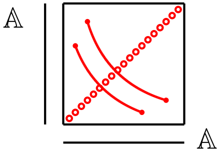

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So for Today's Theorem we only care about varieties that are not locally finite.

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d_t does not have to be idempotent, but it will be idempotent restricted to the range of t on any model.

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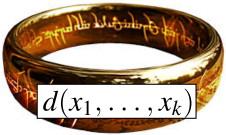

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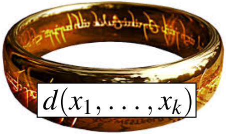

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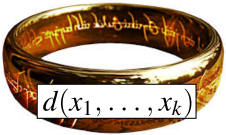

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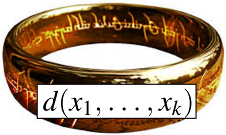

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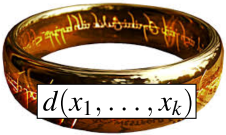

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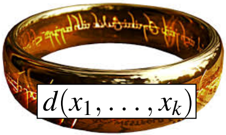

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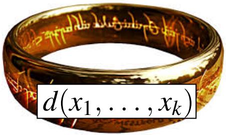

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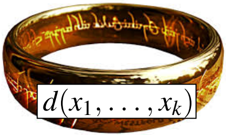

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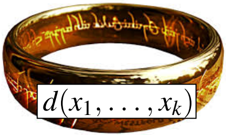

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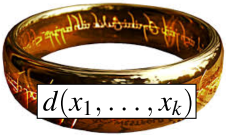

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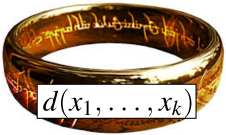

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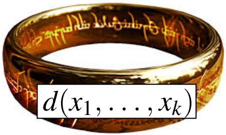

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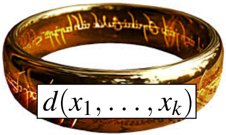

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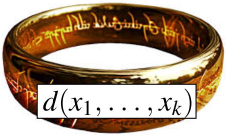

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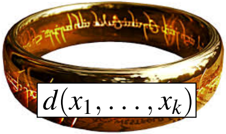

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In particular, the clone of $\mathcal{V}$ is generated by its monoid of unary terms and the operation d.

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The localization functor $\mathcal{V} \mapsto e(\mathcal{V}) : \mathbb{A} \mapsto \langle e(A); \{et(\bar{x})\}\rangle$ is a categorical equivalence from $\mathcal{V}$ to the essentially unary variety $e(\mathcal{V})$.

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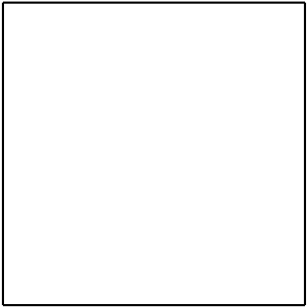

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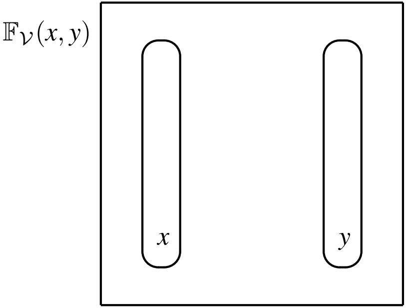

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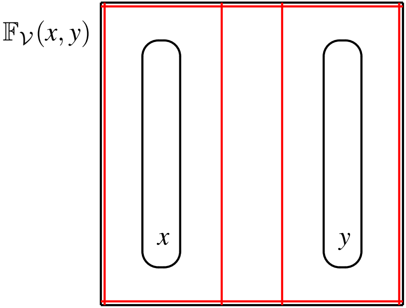

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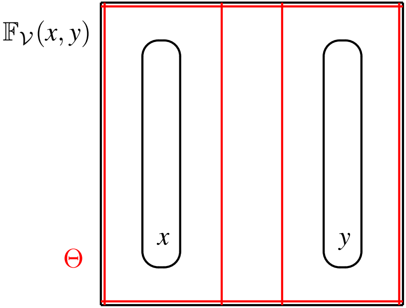

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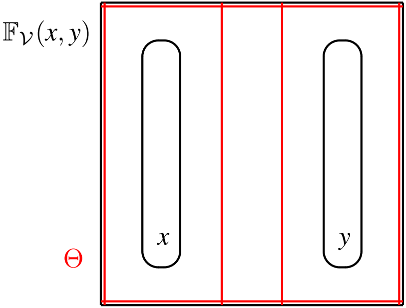

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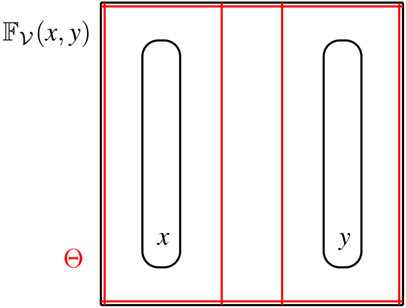

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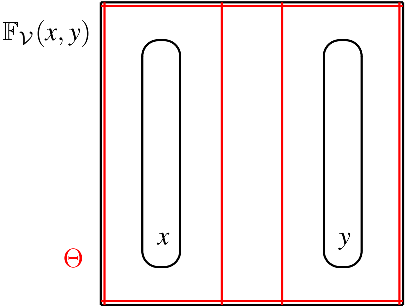

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If $\{a_1, \ldots, a_n\}$ is a partition of unity, then in $\mathcal{B}$ we have

$$(x *_{a_1} (x_2 *_{a_2} (\cdots (x_{n-1} *_{a_{n-1}} x_n) \cdots))) = (a_1 \wedge x_1) \vee \cdots \vee (a_n \wedge x_n)$$