Tensor Product – presentations

Presentations

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Universal Property. (Derived from the universal property of free objects using the First Isomorphism Theorem) There is a set morphism $\iota : G \to \mathbb{P}$ such that, for every set morphism $g : G \to A$ into a $\mathcal{V}$-object $\mathbb{A}$ where $g(G)$ satisfies the relations in R, there is a unique extension of g to an algebra morphism $\widehat{g} : \mathbb{P} \to \mathbb{A}$.

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In general, the difficulty in dealing with $\langle G \mid R\rangle$ is deciding if two elements α, β are equal: $\alpha = w_1(G)/\Theta(R) = w_2(G)/\Theta(R) = \beta$ will hold iff the equality $w_1(G) = w_2(G)$ is provable from the set of relations R.

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In words, $M \otimes_A N$ is the A-module generated by the set $M \times N = \{m \otimes n \mid m \in M, n \in N\}$ of simple tensors, subject to the weakest set of relations needed to make $\otimes$ an A-bilinear operation.

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1. There is a bilinear map $\otimes : M \times N \to M \otimes N : (m, n) \mapsto m \otimes n$, and
2. Any bilinear $g : M \times N \to L$ extends uniquely to an A-linear $\widehat{g} : M \otimes N \to L$.

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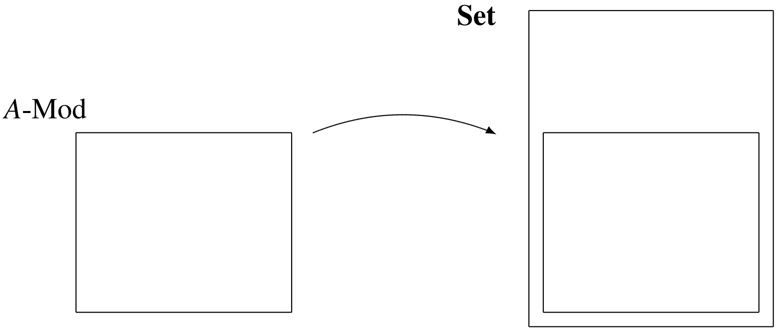
Set
A-Mod

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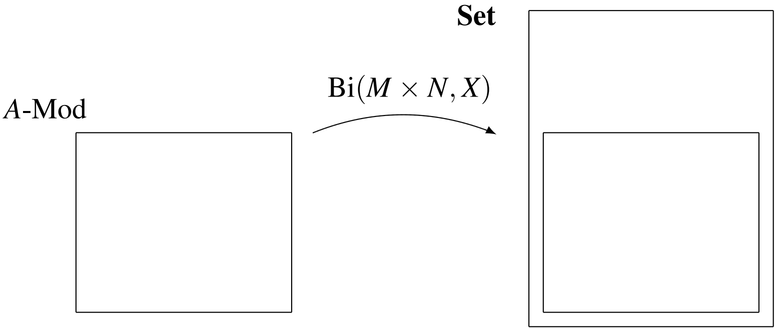

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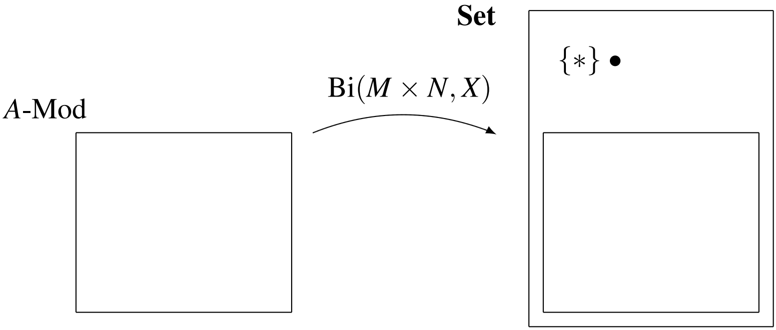

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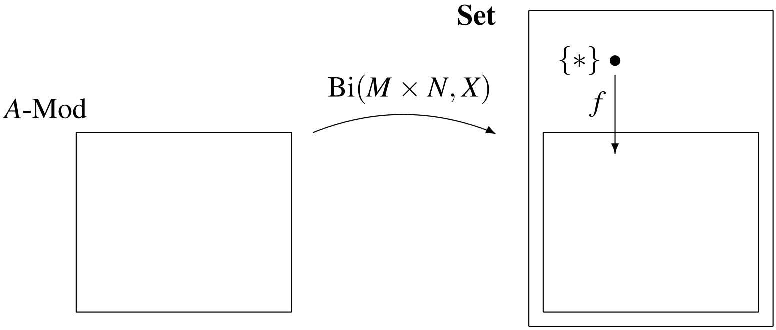

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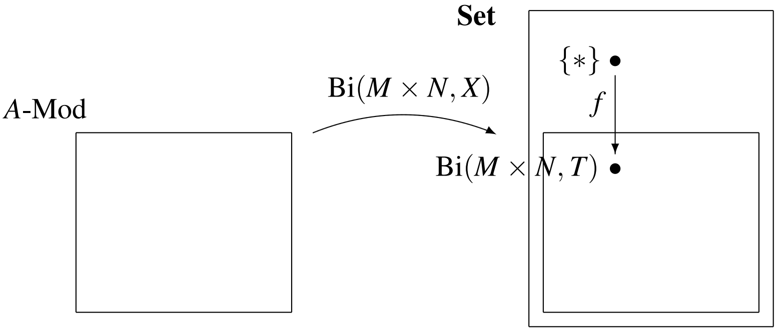

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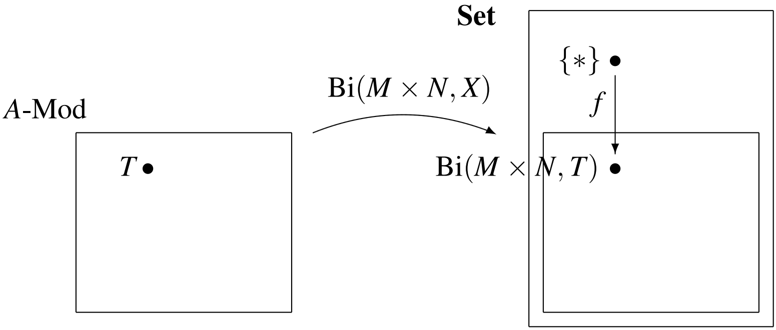

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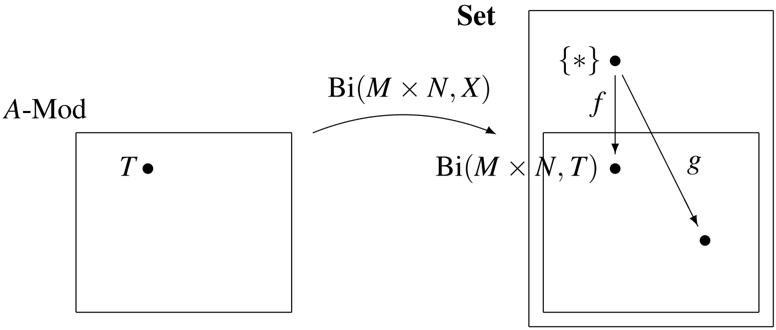

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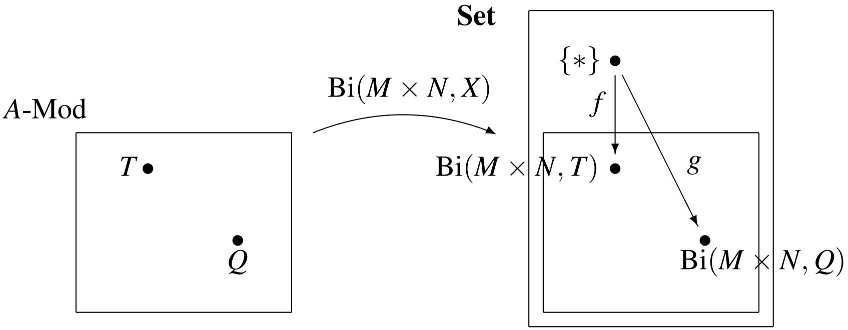

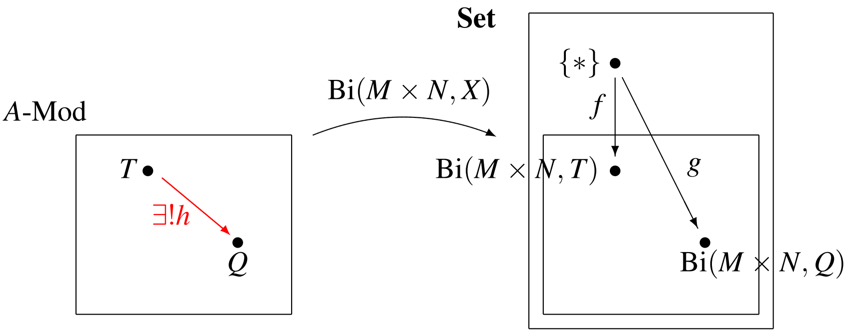
Set
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Bi(M × N, X)
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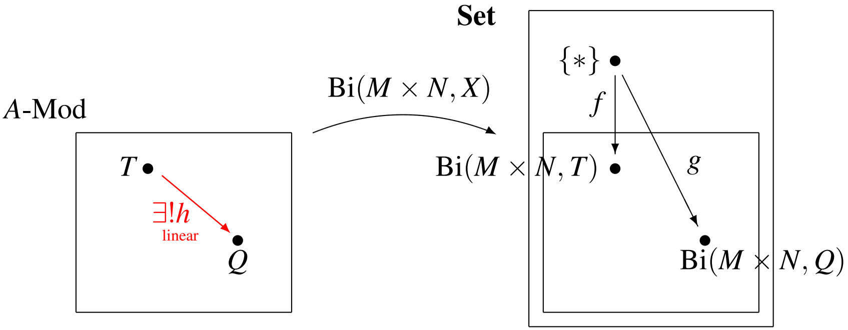

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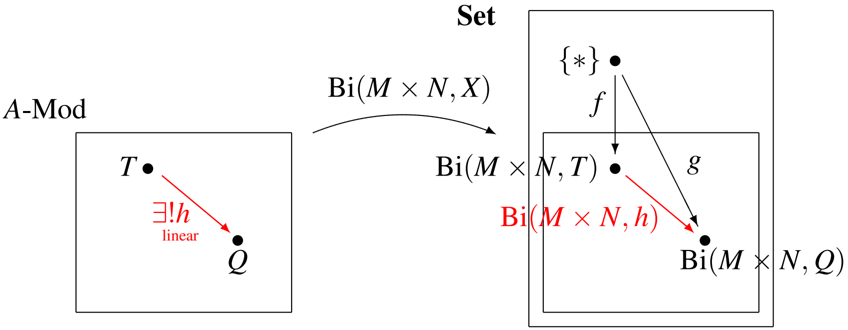

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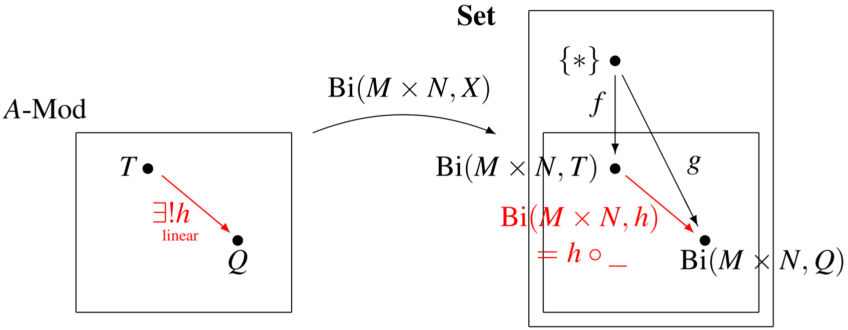

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(Pierre) Mazet's Theorem

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Caracterisation des epimorphismes par relations et generateurs
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