LINEAR ALGEBRA (MATH 3130)
SECTION 2, SPRING 2013
REVIEW SHEET

From the book: Sections 1.1–6.5, excluding Sections 4.8, 5.6–5.8.

I. Systems of linear equations.
   (a) Augmented matrix and coefficient matrix of a system.
   (b) Row reduction. (Reduced) row echelon form. Pivots, pivot positions, pivot columns.
   (c) Free and basic(=pivot) variables. Solutions sets. Parametrized form of a solution.
   (d) Consistent and inconsistent systems.
   (e) Homogeneous systems. Relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of $A\mathbf{x} = \mathbf{0}$.

II. Matrix arithmetic.
   (a) Matrices can be added, negated, multiplied with each other, and scaled, provided the dimensions are right.
   (b) The collection of $n \times n$ real matrices forms an "$\mathbb{R}$-algebra", which is noncommutative if $n > 1$. (This statement indicates which laws of arithmetic are valid for $n \times n$ natrices, namely the ring laws. These laws are enumerated in Theorems 1 and 2 on pages 93 and 97.)
   (d) Matrix transpose.
   (e) Left and right inverses. (Equivalent properties.) Two-sided inverses. A 1-sided invertible matrix is 2-sided invertible iff it is square.
   (f) Elementary matrices. Row reduction is expressible as left multiplication by a sequence of elementary matrices. Matrices A and B are row equivalent iff $A = LB$ for some invertible matrix L.
   (g) Algorithm for finding inverses.
   (h) Partitioned matrices. Block diagonal and block triangular matrices.

III. Vectors and vector spaces.
   (a) Linear systems may be viewed as vector equations.
   (b) Definition of vector space. Definition of subspace.
   (c) Geometric interpretation of vector space operations.
   (d) Definition of column space and nullspace of a matrix.
   (e) Spanning set of vectors.
   (f) Linearly (in)dependent set of vectors.

IV. Linear Transformations.
   (a) Definition. Fact that any linear transformation has the form $T(\mathbf{x}) = A\mathbf{x}$.
   (b) The problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ may be viewed as the problem of finding a vector $\mathbf{x} \in T^{-1}(\mathbf{b})$ for $T(\mathbf{x}) = A\mathbf{x}$.
   (c) One-to-one and onto transformations.
   (d) Finding the standard matrix of a transformation.
   (e) Matrices for rotation and reflection in the plane.

V. Matrix factorization.
   (a) LU factorization.

(b) Solving systems with an LU factorization.

VI. Applications.
(a) Balancing a chemical reaction.
(b) Network flow.
(c) Predator-prey dynamics.
(d) Leontief economic model.

VII. Affine transformations.
(a) An affine transformation has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$. Examples: translations, plane reflections about an arbitrary line, and plane rotations about an arbitrary point.
(b) Affine transformations in $\mathbb{R}^n$ may be represented in homogeneous coordinates in $\mathbb{R}^{n+1}$ by matrices of the form $\left[\begin{array}{c|c} A & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right]$.

VIII. Subspaces
(a) The structure of subspaces of $\mathbb{R}^n$.
(b) Four fundamental subspaces: $\text{Col}(A)$, $\text{Nul}(A)$, $\text{Row}(A) = \text{Col}(A^T)$, and $\text{Nul}(A^T)$.
(c) Ordered and unordered bases for a subspace.
(d) Algorithms for finding bases for $\text{Nul}(A)$, $\text{Row}(A)$, and $\text{Col}(A)$.
(e) Standard basis for $\mathbb{R}^n$.
(f) Dimension of a subspace.
(g) Rank and nullity of a matrix. Rank + nullity theorem.
(h) Proof that "dimension" is well defined, namely, that the size of any independent set is less or equal the size of any spanning set, and that a maximal independent set is spanning while a minimal spanning set is independent.

IX. The determinant.
(a) Signed volume.
(b) Minor, cofactor, definition of the determinant via the Laplace expansion.
(c) $\det(A)$ is defined only if A is square. $\det(A) \neq 0$ iff the columns of A are independent.
(d) Adjugate matrix. Fact that $A \cdot \text{adj}(A) = \det(A) \cdot I$, hence $A^{-1} = (1/\det(A))\text{adj}(A)$ when A is invertible.
(e) Further properties: $\det(AB) = \det(A)\det(B)$, the determinant can be computed by Gaussian elimination, the determinant of a block triangular matrix if the product of the determinants of the blocks, if $T(\mathbf{x}) = A\mathbf{x}$, then the determinant of A measures the "volume expansion" associated with T.
(f) "Correct" definition: the determinant is the unique alternating multilinear function d of n variables defined on $\mathbb{R}^n$ for which $d(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.
(g) Permutation expansion of the determinant. Fact that $\det(A) = \det(A^T)$.
(h) Cramer's Rule for solving a linear system $A\mathbf{x} = \mathbf{b}$ with invertible A.

X. Abstract vector spaces.
(a) Meaning of the word "abstract".
(b) Definition and examples of abstract vector spaces, e.g., $M_{m\times n}(\mathbb{R})$, $\mathbb{P}_n(t)$, $C^k([0,1])$. We computed that $M_{m\times n}(\mathbb{R})$, has dimension mn, $\mathbb{P}_n(t)$ has dimension $n+1$, and that $C^0([0,1])$ must be infinite dimensional.
(c) Coordinates relative to a basis.

(d) Definition of "isomorphism" of vector spaces. Proof that every finitely generated real vector space is isomorphic to $\mathbb{R}^n$ for some finite n.
(e) Matrices, ${}_{\mathcal{C}}[T]_{\mathcal{B}}$, for linear transformations between abstract vector spaces. Change of basis matrices, ${}_{\mathcal{C}}[I]_{\mathcal{B}}$.

XI. Markov chains.
(a) Definitions of: probability vector, (left, right, doubly) stochastic matrix and Markov chain.
(b) Steady state vector.
(c) Regular stochastic matrices have a unique steady state vector that is a probability vector. Its entries are strictly positive.

XII. Eigenvalues, eigenvectors, eigenspaces.
(a) Eigenvectors identify "preserved directions" of a linear transformation $T\colon V \to V$.
(b) Definitions of eigenvector, eigenvalue, eigenspaces.
(c) Methods of calculation: characteristic polynomial $\chi_A(\lambda)$ equals $\det(A - \lambda I)$; e-values of A are the roots of $\chi_A(\lambda) = 0$; e-space V_λ equals $\mathrm{Nul}(A - \lambda I)$; λ-eigenvectors are the nonzero vectors of V_λ. Fast calculation of e-values for (block) triangular matrices.

XIII. Diagonalization.
(a) Structure of roots of a real polynomial over $\mathbb{R}$ or $\mathbb{C}$, and of a complex polynomial over $\mathbb{C}$. Algebraic multiplicity of an e-value.
(b) Geometric multiplicity of an e-value.
(c) Defn. of "diagonalizable". Thm. A transformation $T\colon V \to V$ is diagonalizable iff V has a basis consisting of e-vectors for T iff the geometric multiplicity of each e-value equals its algebraic multiplicity.
(d) Independence of subspaces. Sums of subspaces and direct sums of independent subspaces. A sum of distinct e-spaces is direct. $T\colon V \to V$ is diagonalizable iff $V = \bigoplus_\lambda V_\lambda$. (Side observation: $\dim(U \oplus W) = \dim(U) + \dim(W)$.)
(e) Similarity: A is similar to B if A is a conjugate of B, i.e., $A = S^{-1}BS$. Similarity is an equivalence relation on the set of $n \times n$ matrices. Matrices are similar iff they represent the same transformation relative to different bases. Similar matrices have the same characteristic polynomial, hence same e-values. If $A = S^{-1}BS$, then $S\colon V_\lambda^A \to V_\lambda^B$ is an isomorphism for each e-value λ. A is diagonalizable iff it is similar to a diagonal matrix.
(f) (The nondiagonalizable case.) Generalized e-spaces $V_\lambda^{(\infty)} = \bigcup_k \mathrm{Nul}(A - \lambda I)^k$.
(g) The algebraic multiplicity of λ is $\dim(V_\lambda^{(\infty)})$.
(h) If $T\colon V \to V$ is defined over $\mathbb{C}$ and V is f.g., then V is the direct sum of its generalized e-spaces.
(i) Jordan block, Jordan canonical form. Eigenchains in $V_\lambda^{(\infty)}$ yield JCF. Every transformation defined over $\mathbb{C}$ is similar to a matrix in JCF, and the JCF is unique up to a permutation of Jordan blocks.
(j) The JCF of A can be determined indirectly from the numbers 'nullity$(A - \lambda I)^k$' for all λ and k.
(k) Diagonalization and JCF of T over $\mathbb{R}$ instead of $\mathbb{C}$: $\overline{V_\lambda^{(\infty)}} = V_{\overline{\lambda}}^{(\infty)}$, and $V_\lambda^{(\infty)} \oplus V_{\overline{\lambda}}^{(\infty)}$ has a nice real basis consisting of the real and imaginary parts of the vectors in the

λ-eigenchains in $V_\lambda^{(\infty)}$. This choice of basis results in diagonal form or JCF with 2×2 real blocks replacing pairs of 1×1 conjugate complex blocks.

XIV. Orthogonality.
   (a) Dot product. (Defn. Arithmetic facts follow from those of matrices, since $\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T \mathbf{v}$.)
   (b) Length in $\mathbb{R}^n$. Unit vector in direction $\mathbf{v}$ is $\mathbf{v}/\|\mathbf{v}\|$.
   (c) Angle in $\mathbb{R}^n$ via $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$. Cauchy-Schwarz Inequality.
   (d) Orthogonality. Orthogonal complement. $\text{Row}(A)^\perp = \text{Nul}(A)$. Algorithm for finding the orthogonal complement of a set of vectors.
   (e) Orthonormal set of vectors. Angle-preserving linear transformations. Orthogonal matrices.
   (f) Approximate solutions to $A\mathbf{x} = \mathbf{b}$ via least squares. Normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$. Fitting curves to data.
   (g) Orthogonal projection onto a vector or subspace. Gram-Schmidt algorithm.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Chapter 1–6 supplementary problems (excluding problems marked [**M**]).

(2) Let A be a square matrix. Explain why if the columns of A are independent, then the columns of A^2 are independent.

(3) Show that if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, then $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}+\mathbf{c}\}$ is also independent.

(4) Explain why if A and B are $n \times n$ matrices satisfying $A\mathbf{x} = B\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

(5) Use the definition of "linear transformation" to show that the composition of two linear transformations is a linear transformation.

(6) Among all $n \times n$ matrices whose entries are all either 0 or 1 what is the maximum possible number of 1's if the matrix is invertible? What is the minimum number of 1's if the matrix is invertible? For which values of n is it possible for the number of 1's to be equal to the number of 0's and still have the matrix invertible?

(7) Explain why if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subset of $\mathbb{R}^n$ that is linearly independent and spans the space, then $k = n$.

(8) Which matrices commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$? (Hint: set up a linear system.)

(9) Explain why the set of columns of an $n \times n$ invertible matrix spans $\mathbb{R}^n$. Then explain why this set of columns is independent.

(10) Explain why every $n \times n$ matrix M is expressible in exactly one way as a sum $M = S + A$ where S is symmetric and A is antisymmetric.

(11) Show that if A is invertible, then A has at most one LU factorization.

(12) Give an example of a square matrix with no LU factorization.

(13) Computational problems:

(a) Using homogeneous coordinates, find a matrix representation for the transformation that rotates the plane 45° around the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Find bases for the null space, row space and column space of the 3×3 matrix whose entries are all 1. What are the dimensions of these spaces?

(c) Put the numbers $1, 2, \ldots, 9$ into a 3×3 matrix in order. What is the determinant?

(d) Find a change of basis matrix from $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ to $\mathcal{C} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(e) Find the characteristic equation, e-values, and e-spaces of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a matrix S that conjugates A into diagonal form.

(f) Find a basis for $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}^{\perp}$.

(14) Can any of the following exist? (If so, give an example, if not give a reason.)

(a) A vector space with an empty basis.
(b) A matrix of rank zero.
(c) A matrix with no determinant.
(d) A matrix with a zero dimensional eigenspace.
(e) An invertible matrix whose row sums are all zero.
(f) A real matrix whose null space equals its column space.
(g) A matrix A such that $\text{nullity}(A) = 1$ and $\text{nullity}(A^2) = 3$.
(h) A matrix where the dimension of the row space is greater than the dimension of the column space.
(i) A real number that does not arise as the determinant of a real matrix.
(j) A vector space with no subspaces.
(k) An isomorphism between vector spaces of different dimensions.
(l) A matrix whose row space is isomorphic to its column space.
(m) A matrix whose characteristic polynomial is $\lambda^2 + \lambda + 1$.

(n) An eigenvalue whose geometric multiplicity exceeds its algebraic multiplicity.
(o) A real 10×10 matrix with only one eigenvector.
(p) A matrix equal to its adjugate.
(q) A left stochastic matrix that is not right stochastic.
(r) Conjugate matrices of different ranks.
(s) Conjugate matrices that are not similar.
(t) A matrix that is diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.
(u) An orthogonal basis for $\mathbb{R}^3$ that is not orthonormal.
(v) A nondiagonalizable complex matrix.
(w) An orthogonal matrix with determinant zero.
(x) A 3×3 orthogonal matrix with no zero entries.
(y) Subspaces U and W such that $U + W \neq U \oplus W$.
(z) A real vector that is orthogonal to itself.

(15) Give the dimensions of the following real vector spaces.
 (a) The space of real polynomials $p(t)$ of degree at most 3 which satisfy $p(1) = p(-1) = 0$.
 (b) The space of 3×3 upper triangular real matrices.
 (c) The space of twice continuously differentiable functions $y = f(x)$ satisfying $y'' = 0$.

(16) How would you solve the following problem? Suppose that V has basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and U is a subspace of V with basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. How do you find a basis for V whose first m vectors form a basis for U?

(17) Suppose that you are given bases $\mathcal{B}$ and $\mathcal{C}$ for subspaces U and W of a space V. How would you find a basis for $U + W$? How would you find a basis for $U \cap W$? (Hint: in both cases, you should apply Gaussian Elimination to the matrix $[\mathcal{B}|\mathcal{C}]$. How should you use the results?)

(18) Is there a 3×3 matrix whose minors are nonzero and all equal? Is there a 3×3 matrix whose cofactors are nonzero and all equal?

(19) Let S be a 2×2 invertible matrix. Consider the linear transformation of "conjugation by S":

$$T \colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \colon A \mapsto S^{-1}AS.$$

Show that if λ is an e-value of T, then so is λ^k for any k. Show that 0 is not an e-value of T. Explain why the e-values of T can only be $+1$ or -1. Show that $+1$ occurs as an e-value with multiplicity at least 2.

(20) What is the characteristic polynomial for the $n \times n$ matrix whose entries are all 1?

(21) Show that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. (Solution 1 hint: choose a basis for $U \cap W$ and extend it in different ways to bases for both U and W. Show that all the vectors together form a basis for $U + W$.) (Solution 2 hint: let $\mathcal{B}$ and $\mathcal{C}$ be bases for U and W. Apply the rank+nullity theorem to the matrix $[\mathcal{B}|\mathcal{C}]$.)

(22) The points $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the vertices of a regular tetrahedron. Find the lengths of the sides and the angles formed by adjacent faces.

(23) Find the least squares curve of the form $y = ax^2 + bx + c$ that best fits the data points $(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)$.

(24) Show that if V is finite dimensional and U is a subspace, then $V = U \oplus U^\perp$.

(25) Show that $(U + W)^\perp = U^\perp \cap W^\perp$.

(26) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$.