DISCRETE MATH (MATH 2001): REVIEW SHEET

I. Set Theory
   (a) Informal notion of a set. The axioms.
   (b) Valid constructions of new sets (pairing, union, power set, comprehension, intersection)
   (c) Empty set, successor of a set.
   (d) Inductive sets, natural numbers.
   (e) Russell's Paradox.

II. Logic
   (a) Formulas
      (i) Alphabet of symbols: variables, equality, connectives, quantifiers, predicate symbols, punctuation symbols.
      (ii) Terms, atomic formulas, formulas and sentences.
      (iii) Formula trees, term trees.
   (b) Propositional logic
      (i) Truth tables.
      (ii) Tautologies, contradictions, logical equivalence.
      (iii) Contrapositive and converse.
      (iv) Equivalence of $(H \to C)$, $((\neg C) \to (\neg H))$, and $((H \wedge (\neg C)) \to \text{False})$.
      (v) Converse. Relationship between $(H \to C)$ and $(C \to H)$.
      (vi) Disjunctive normal form.
   (c) Structures (definition and examples).
   (d) Truth of a sentence in a structure.
      (i) Converting a sentence to prenex form (including: scope of a quantifier, free and bound variables, rules for changing the order of quantifiers and connectives).
      (ii) Quantifier games to determine the truth of a sentence in prenex form in a given structure.
      (iii) The consequence relation, $\models$. Valid sentences.
   (e) Proof.
      (i) Definition of proof. The provability relation, $\vdash$.
      (ii) Accepted laws of deduction are modeled on $\models$.
      (iii) Statement of the completeness theorem.
      (iv) Methods of proof. (Direct proof, proof of the contrapositive, proof by contradiction.)
      (v) Proof writing strategies. (Proof versus disproof. Direct versus indirect proof. "If and only if." Proving the equivalence of several statements. Case division. Treatment of quantifiers. Examples versus counterexamples.)

III. Induction
  (a) Ordinary induction.
  (b) Strong induction.
  (c) Recursive definitions of arithmetic operations on $\mathbb{N}$: $x + y, xy, x^y$.
  (d) Use of induction to prove laws of arithmetic.

IV. Relations
  (a) Ordered pairs, triples and n-tuples. $A \times B$.
  (b) Definition of a function. Representations of functions. Composition.
  (c) Injections, surjections, bijections.
  (d) Canonical factorization of a function.
  (e) Kernel of a function. Equivalence relations. Partitions. Relationships between these three.
  (f) Posets: strict and nonstrict orders. Extensions. Linear orders.

V. Constructing the integers from the natural numbers
  (a) The natural numbers are well ordered.
  (b) Definition of the integers.
  (c) Definition of the arithmetic and order of the integers.
  (d) Meaning of 'well-defined'. Proofs that the arithmetic operations and order on the integers are well-defined.
  (e) Definition of homomorphism. Construction of a 1-1 homomorphism of the natural numbers into the integers.

VI. Counting
  (a) Definitions of cardinality, $|A| = |B|$, $|A| \leq |B|$, $|A| < |B|$, $|A| = m$, finite, infinite, countable, uncountable.
  (b) Proof of the Sum Rule and Product Rule.
  (c) Simple counting formulas:
    (i) There are n^m functions from an m-element set to an n-element set.
    (ii) There are 2^n characteristic functions on an n-element set, and 2^n subsets of an n-element set.
    (iii) There are $n!$ bijections from an n-element set to an n-element set. There are $n!$ ways to linearly order an n-element set.
    (iv) There are $n!/(n-m)!$ injections from an m-element set to an n-element set. There are $n!/(n-m)!$ ways to to choose and linearly order m elements from an n-element set. The value $n!/(n-m)!$ is sometimes denoted $(n)_m$, $P(n, m)$, $n^{\underline{m}}$, etc.
    (v) There are $\binom{n}{m} = C(n, m) = \frac{n!}{m!(n-m)!}$ different m-elements subsets of an n-element set.
  (d) Binomial and multinomial coefficients: binomial theorem, Pascal's identity, Pascal's triangle. Multinomial theorem, Pascal's identities for multinomial coefficients, Pascal's pyramid.

(e) Principle of inclusion and exclusion: formula, counting surjective functions, counting derangements.
(f) Stirling numbers and Bell numbers. Recursion and formula for Stirling numbers. Binomial-type theorem for Stirling numbers.
(g) Distribution problems.
(h) Discrete probability: sample space, event, probability distribution, uniform distribution.

VII. Graph theory
(a) Definitions of: graph, multigraph, directed graph, adjacency, incidence.
(b) Examples: paths, cycles, complete graphs, complete r-partite graphs, the Petersen graph.
(c) Planar drawings and planar graphs.
(d) Euler's Formula: $v - e + r = 2$.
(e) Characterizations of bipartite graphs.
(f) Edge bounds for loopless planar graphs with enough vertices ($e \leq 3v - 6$ if $3 \leq v$; $e \leq 2v - 4$ if $4 \leq v$ and graph is bipartite).
(g) Kuratowski's characterization of planar graphs.
(h) Euler characteristic of a compact 2-manifold.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.
(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.
(1) Explain why $2 + 2 = 4$.
(2) Show that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.
(3) Explain why induction is a valid form of proof. (Your explanation should make use of the fact that $\mathbb{N}$ is a subset of every inductive set.)
(4) Prove that $m(n + k) = (mn) + (mk)$ for all $m, n, k \in \mathbb{N}$.
(5) Write a formal sentence expressing the axiom of union. Then draw a formula tree for your sentence.

(6) Put $((p \to q) \to r)$ in disjunctive normal form.

(7) Describe a winning strategy for either $\exists$ or $\forall$, which determines the truth of
$$\forall x\, \forall y\, \exists z\, ((x < y) \to ((x < z) \wedge (z < y)))$$
in (i) $\langle \mathbb{R}; < \rangle$, (ii) $\langle Z; < \rangle$.

(8) Put $(A \leftrightarrow \forall x\, B(x))$ in prenex form. You may assume that A has no free variables.

(9) What is a function? (Give the definition.)

(10) What is the kernel of the squaring function $F\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$?

(11) How many different partitions are there on the set $\{1, 2, 3, 4\}$?

(12) Draw an example of an ordered set that has 1 minimum element and 5 maximal elements. Then draw one that has 0 minimum elements, 1 minimal element and 5 maximal elements.

(13) Describe the procedure for constructing the set of integers from the set of natural numbers. Without proving anything, identify the statement that must be proved to verify that the procedure works.

(14) Show that the integers satisfy $\forall x\, \forall y\, (x + y = y + x)$.

(15) If $x = [(k, \ell)]_E$ and $y = [(m, n)]_E$, then set $x * y = [(k \cdot m, \ell \cdot n)]_E$. Is this a well-defined operation on $\mathbb{Z}$?

(16) How many ways are there to make a circular necklace with n beads of different colors if two necklaces are considered to be the same if they differ by a rotation? What if two necklaces are considered to be the same if they differ by a rotation or a flip?

(17) What is the constant term in $(x^{-2} + 2x^{-1} + 3 + 5x)^3$?

(18) You have just given birth to octuplets. How many ways can you name your children if you only like the names Billy Bob, Jim Bob and Sue Bob?

(19) If you deal a random 2-card hand, what is the probability of blackjack? (An ace together with a 10 or face card.)

(20) How many loopless multigraphs with vertex set $\{v_1, \ldots, v_n\}$ have k edges? What if loops are allowed?

(21) Let C_n be a cycle of length n and let K be a set of k colors. How many proper colorings of C_n are there which use only colors from K?

(22) Consider a graph to be a structure $G = \langle V; E \rangle$ where E is a binary predicate on the set V. Thus $E(a, b)$ holds if vertices a and b are connected by an edge. Write formal sentences that hold in G iff
  (a) any two vertices are connected by a path of length 3.
  (b) K_4 is a subgraph.
  (c) the diameter is 2.

(23) Write a formal sentence that distinguishes between the Petersen graph and K_5.

(24) Does the Petersen graph have a Hamiltonian cycle?

(25) Give a simple description of the class of graphs that satisfy the following sentence.

$$\forall x \, \exists y \, \forall u \, \exists v \, (E(x,y) \wedge E(y,u) \wedge E(u,v) \wedge E(v,x))$$

(26) Find the Euler characteristic of the 2-holed torus.