Math 3130, Introduction to Linear Algebra

Catalog description: Examines basic properties of systems of linear equations, vector spaces, linear independence, dimension, linear transformations, matrices, determinants, eigenvalues, and eigenvectors. Prereq., MATH 2300 or APPM 1360.[1] Credit not granted for this course and APPM 3310.[2]

Math 3130, Introduction to Linear Algebra, is a course designed to prepare students for a wide range of courses in mathematics, physics, economics, computer science and engineering. The course is taught with a mixture of theory and application. Students are expected to leave the course able to solve problems using linear algebra, to know a number of applications of linear algebra, to be able to follow complex logical arguments and to develop simple logical arguments.

The chief theoretical topics and their objectives are:

(1) Systems of linear equations.
- (a) Set up and solve a system of linear equations using Gaussian elimination.
- (b) Be able to distinguish between consistent and inconsistent systems.
- (c) Relate the solutions of a system to the solutions of the associated homogeneous system.
- (d) Rewrite a linear system as a matrix equation or a vector equation.

(2) Matrices and matrix algebra
- (a) Be able to identity laws of matrix algebra, and to give examples of matrices satisfying $AB \neq BA$.
- (b) Explain the laws of transpose. Define symmetric and antisymmetric matrices.
- (c) Explain left, right and 2-sided invertibility. Be able to link each invertibility property for A to the corresponding solvability property of the system $A\mathbf{x} = \mathbf{b}$.
- (d) Define *elementary matrices* and explain their role in Gaussian elimination.
- (e) Define *(block) diagonal* and *(block) triangular* matrices.

(3) Vector spaces
- (a) Explain the defining properties of vectors spaces and give examples of vector spaces.
- (b) Explain the geometric interpretation of the vector space operations in $\mathbb{R}^n$.
- (c) Define the four fundamental subspaces associated with a matrix.
- (d) Define *linear combination*, *linearly independent*, *span*, *basis*, and *dimension.*
- (e) Explain how to write a vector in coordinates relative to a basis.
- (f) Compute the dimension of any of the four fundamental subspaces.
- (g) Define *rank* and *nullity* and prove the rank+nullity theorem.

(4) Linear transformations
- (a) Define *linear transformation.*
- (b) Explain why any linear transformation between finitely generated spaces has a matrix representation.
- (c) Compute the matrix of a transformation relative to specified input and output bases.
- (d) Define *image* and *kernel*, and use these words in definitions of *onto* and *1-1* transformations. Explain the link between (image, kernel) and (rank, nullity). Define *isomorphism.*

[1] Calculus 2.

[2] Matrix Methods.

   (e) Explain why any finite dimensional real vector space is isomorphic to $\mathbb{R}^n$ for some n.
(5) Determinant
   (a) Explain the connection between the determinant and area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$.
   (b) Describe the cofactor expansion of the determinant and be able to use it to compute some determinants.
   (c) Be able to compute the inverse of a matrix via the determinant formula for inverse.
(6) Eigenvalues and eigenvectors
   (a) State the definitions of *eigenvalue*, *eigenvector*, and *eigenspace*.
   (b) Given a square matrix, find its characteristic polynomial, eigenvalues and bases for its eigenspaces.
   (c) State necessary and sufficient conditions for a matrix to be diagonalizable.
   (d) Find a diagonalizing matrix, if one exists.
   (e) Define similarity, and explain how the characteristic polynomial, eigenvalues and eigenspaces of similar matrices are related.
   (f) Explain how to raise a diagonalizable matrix to a large power, and how to use this to solve linear recurrences.
(7) Orthogonality
   (a) Explain the concept of orthogonality of vectors and subspaces.
   (b) Explain the orthogonality relations among the four fundamental subspaces.
   (c) Given $\mathbf{u}$ and $\mathbf{v}$, compute $\|\mathbf{u}\|$, the unit vector in the direction of $\mathbf{v}$, the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (d) State the Cauchy-Schwarz Theorem.
   (e) Construct a basis for the orthogonal complement of a set of vectors in $\mathbb{R}^n$.
   (f) Given a least-squares problem, derive the normal equations and solve. Be able to apply this to fit a curve to data.
   (g) Be able to apply the Gram-Schmidt process to construct an orthonormal set of vectors.
   (h) Given a vector $\mathbf{u}$ and a subspace W of $\mathbb{R}^n$, compute the orthogonal projection of $\mathbf{u}$ onto W.
   (i) Be able to define orthogonal transformations and to give examples.

There are many types of applications that could be covered in a semester course. Fitting polynomial curves to data, network flow and electrical network problems are good for illustrating the practicality of linear systems, and for gaining practice with Gaussian elimination. Discrete dynamical systems and Markov chains, where one might look for a steady state vector, could be used to lead into a discussion of eigenvectors. Linear recurrence relations illustrate the usefulness of diagonalizing. Any application can be used to illustrate the method of least squares.