Math 2001

Math 2001, Introduction to Discrete Mathematics, is a course designed to prepare students for upper division mathematics classes. Students are expected learn standard mathematical terminology, foundational material, and methods of proof, all within the context of discrete mathematics.

Essential topics, which ought to span 1/2—2/3 of a semester, are

(1) sets,
(2) relations,
(3) logic, and
(4) proof,

not necessarily in this order. Secondary topics are often chosen from combinatorics, graph theory, ordered sets, discrete probability, game theory, or discrete geometry.

Set theory. Minimally, students should hear that mathematics is founded on set theory, which is usually formulated by a short list of axioms. Students should learn the meanings of the symbols $\in$, $\emptyset$, $\cup$, $\cap$, $\mathcal{P}(X)$, as well as the notation $A = \{0, 1, 2\}$ and $A = \{x \in B \mid P(x) \text{ holds}\}$. Students should learn how to show that $A = B$, $A \neq B$, $A \in B$, $A \notin B$, $A \subseteq B$ or $A \nsubseteq B$.

Maximally, it is also useful for students to learn that each natural number equals the set of its predecessors and that $\mathbb{N}$ is defined to be the smallest set containing 0 and closed under successor. (This is useful when discussing induction.)

Relations. Minimally, students should learn what an ordered pair is, what $A \times B$ means, and how relations are defined. They should learn what a function is, what *injective*, *surjective* and *bijective* mean. They should learn what equivalence relations are, and that they are kernels of functions. They should learn how to determine if a function is well-defined. They should learn that mathematical definitions are used to make concepts definite.

Maximally, students ought to learn that ordered pairs, cartesian products, relations and functions are all sets, and that two such objects are equal if they are equal as sets. They should learn that composition of functions is associative. They should learn about the relationship between equivalence relations and partitions. They should learn how to define functions by recursion (e.g., sum and product of natural numbers). They should learn what $|A| = |B|$, $|A| \leq |B|$, $|A| < |B|$, *finite* and *infinite* mean.

Logic. Minimally, students should hear that most of mathematics concerns declarative statements, which can be combined with the conjunctions *and*, *or*, *not*. They should learn how a composite statement depends on its constituents. They should learn about quantifiers, and how to use them. They should be able to read and write simple formal statements, and to translate back and forth between formal and informal statements. They should be able to negate a formal statement.

Maximally, students should learn about logical equivalence. It may help to discuss formula trees. They should learn how to interpret restricted quantifiers, like $(\forall x P(x))\ (Q(x))$ and $(\exists x P(x))\ (Q(x))$. (Concrete example: $(\forall x > 1)(x^2 > x)$.)

Proof. Minimally, students should hear that a proof is a sequence of statements in which each statement follows from preceding statements via accepted rules of deduction. Students should learn the methods of direct proof, proof of the contrapositive and proof by contradiction. Students should be able to prove a simple statement by any of these methods. They should learn how a statement differs from its converse. Students should learn (and practice) induction.

Maximally, students ought to learn why direct proof, proof of the contrapositive and proof by contradiction are equivalent. Students ought to learn how to determine if a simple statement is true in a structure. (E.g. is $\forall x\ \exists y\ (x > y)$ true in $\mathbb{R}$? What about $\mathbb{N}$?) They should learn why induction is a valid form of proof.

Sample questions. At the end of the course, students ought to be able to understand and solve problems like these.

(1) Find sets A and B such that $A \in B$ and $A \subseteq B$, or prove that there are no such sets. Then find sets C and D such that $C \notin D$ and $C \subseteq D$, or prove that there are no such sets.

(2) Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$.

(3) A theorem with hypothesis H and conclusion C has the form $H \to C$. Suppose that you can prove that the converse of the theorem implies the contrapositive of the theorem. Does this help you to prove the theorem?

(4) Express the negation of $(\forall x > 0)(\exists y(y^2 = x))$ in a form where the quantifiers are in front.

(5) Is $\forall w\ \exists x\ \forall y\ \exists z\ (w^2+x^2 = y^2+z^2)$ a true statement about the real numbers? Explain.

(6) Prove or disprove: (a) If $f, g\colon \mathbb{R} \to \mathbb{R}$ are both increasing, then $f \circ g$ is increasing. (b) If $f, g\colon \mathbb{R} \to \mathbb{R}$ are both surjective, then $f \circ g$ is surjective. (c) If $f, g\colon \mathbb{R} \to \mathbb{R}$ are both bounded above, then $f \circ g$ is bounded above.

(7) Let E be an equivalence relation on X. Under what circumstances is $f\colon X/E \to X\colon [x]_E \mapsto x$ well-defined?

(8) Show that $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} < 2\sqrt{n}$.

(9) Show that $(3/2)^n \leq F_{n+2} \leq 2^n$, where F_n denotes the nth Fibonacci number.

(10) Show by induction that every positive integer is a product of primes.