An Easy Test For Congruence Modularity

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CU Logic Seminar

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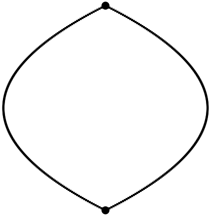

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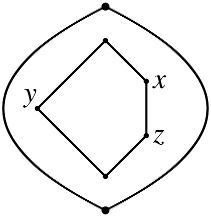

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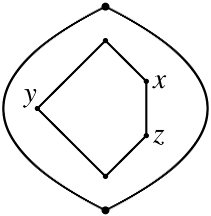

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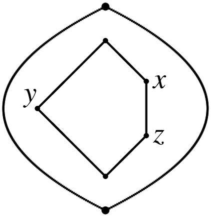

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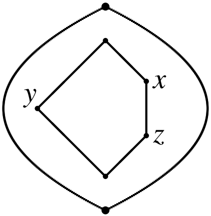

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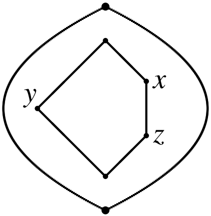

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The modular law was discovered by Richard Dedekind as a law satisfied by the subgroup lattice of $\langle \mathbb{C}; +, -, 0 \rangle$. He also proved that any law in at most 3 variables satisfied by the subgroup lattice of $\mathbb{C}$ is a consequence of the modular law.

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Starting in the 1970's, commutator theory was developed for congruence modular varieties. This theory has been applied to problems about counting models, decidability problems, the study of categorical algebraic properties of varieties, finite basis questions, equational completeness, and more.

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$$m_0(x,u,v,y) = x, \quad m_1(x,u,v,y) = xv^{-1}ux^{-1}y, \quad m_2(x,u,v,y) = y.$$

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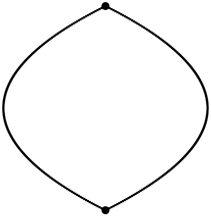

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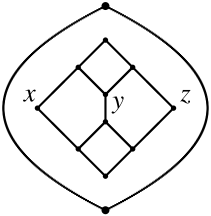

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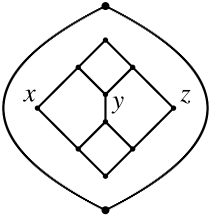

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Bentz suggested a possible counterexample to "only if". He showed that if you define a variety with identities guaranteeing $T_0 \Rightarrow T_1$ and add to those $x \approx q_1(x,y,y)$, $q_1(x,x,y) \approx q_2(x,x,y)$, $q_2(x,y,x) \approx x$, $q_2(x,y,y) \approx p(x,y,y)$, and $p(x,x,y) \approx y$, then topological algebras must satisfy $T_0 \Rightarrow T_2$. These are the Gumm identities for congruence modularity for $n = 2$ *minus* the Gumm identity $q_1(x,y,x) \approx x$.

Question. Do the Gumm identities for $n = 2$ imply congruence modularity if you delete the Gumm identity $q_1(x,y,x) \approx x$?

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Note: If Σ is idempotent and F is independent of its first place, then

$$\Sigma \models F(y, x, \ldots, x) \approx F(x, x, \ldots, x) \approx x,$$

so F is weakly independent of its first place.

The derivative of a set of identities

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Example. If $\Sigma = \{F(x, y, y) \approx x\}$, then

$$\Sigma' = \{F(x, y, y) \approx x, F(x, y, z) \approx F(x, y', z), F(x, y, z) \approx F(x, y, z')\}$$

where $xyy'zz'$ are all distinct.

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From identities (a), (c) and (d), infer

$$\Sigma' \models x \overset{(a)}{\approx} m_0(x,*,*,y) \overset{(d)}{\approx} m_1(x,*,*,y) \overset{(c)}{\approx} \cdots \overset{(?)}{\approx} m_n(x,*,*,y) \overset{(a)}{\approx} y.$$

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Note: This shows that Gumm's identities are slightly redundant.

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C is fully independent of all places relative to Σ', so

$$\Sigma' \models x \approx C(x, x, \ldots, x) \approx C(y, y, \ldots, y) \approx y.$$

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(This does not contradict anything in the talk.)