Imaginaries

Let L be a first-order language and A an L-structure. We define an expansion of L to a new language L_A^{Eq}. One can view this expansion as the usual first-order equivalent of an expansion of L to a certain many-sorted language; more on this below.

Let Eq be the set of all triples $e = (\varphi, \overline{x}, \overline{y})$ such that φ is a formula of L, $\overline{x}$ and $\overline{y}$ are disjoint sequences of distinct variables of the same length, every free variable of φ is in one of these lists, and $\{(\overline{a}, \overline{b}) : A \models \varphi(\overline{a}, \overline{b})\}$ is an equivalence relation on some nonempty set ∂_e. The equivalence class of a tuple $\overline{a}$ is denoted by $\overline{a}/e$. If $e \in \mathrm{Eq}$, we write $e = (\theta^e, \overline{x^e}, \overline{y^e})$. The common length of $\overline{x^e}$ and $\overline{y^e}$ is denoted by $\mathrm{len}(e)$. A special role is played by the member $(x = y, x, y)$ of Eq; we denote it by $\mathfrak{i}$.

The new non-logical symbols of L^{eq} are as follows:

(1) For each $e \in \mathrm{Eq}$, a 1-ary relation symbol P_e.

(2) For each $e \in \mathrm{Eq}$, a function symbol $\mathscr{F}_e$ of rank $\mathrm{len}(E)$.

These symbols P_e act as the translations from a many-sorted expansion of L. As part of this translation, we also fix an element $a \in A$ and let $u = (a/\mathfrak{i}, \mathfrak{i})$. Our expansion depends on a, but in an inconsequential way, since u is used as a value of function symbols when we are not interested in what happens under its arguments.

Now we define an L^{eq}-structure A^{eq} as follows. Its universe is the set of all pairs $(\overline{a}/e, e)$ such that $e \in \mathrm{Eq}$ and $\overline{a} \in \partial_e$. For any $e \in \mathrm{Eq}$, the function $f_e : \partial_e \to A^{\mathrm{eq}}$ is defined by $\overline{a} \mapsto (\overline{a}/e, e)$. Let $I_e = \{(\overline{a}/e, e) : \overline{a} \in \partial_e\}$. So $f_e : \partial_e \to I_e$. Note that $\partial_{\mathfrak{i}} = A$, and $f_{\mathfrak{i}}(a) = (a/\mathfrak{i}, \mathfrak{i})$ for any $a \in A$. Thus $f_{\mathfrak{i}}$ is a one-one function. The denotations of the symbols of L_A^{eq} are determined as follows.

(1) If $\mathbf{c}$ is an individual constant of L, then $\mathbf{c}^{A^{\mathrm{eq}}} = f_{\mathfrak{i}}(c^A)$.

(2) If $\mathbf{R}$ is an m-ary relation symbol of L, then

$$\mathbf{R}^{A^{\mathrm{eq}}} = \{(f_{\mathfrak{i}} \circ \overline{a} : \overline{a} \in \mathbf{R}^A\}.$$

(3) If $\mathbf{F}$ is an m-ary function symbol of L, then

$$\mathbf{F}^{A^{\mathrm{eq}}}(\overline{a}) = \begin{cases} f_{\mathfrak{i}}(\mathbf{F}^A(\overline{b})) & \text{if } \overline{a} = f_{\mathfrak{i}} \circ \overline{b}, \\ u & \text{if there is no such } \overline{b}. \end{cases}$$

(4) $$P_e^{A^{\mathrm{eq}}} = I_e.$$

(5) $$\mathscr{F}_e^{A^{\mathrm{eq}}}(\overline{a}) = \begin{cases} f_e(\overline{b}) & \text{if } \overline{a} = f_{\mathfrak{i}}(\overline{b}), \\ u & \text{otherwise.} \end{cases}$$

Lemma 1. *A is isomorphic to a relativized reduct of A^{eq}. Namely $P_{\mathfrak{i}}^{A^{\mathrm{eq}}}$ is closed under each operation $\mathbf{F}^{A^{\mathrm{eq}}}$ for $\mathbf{F}$ a function symbol of L, and if we let $B = (P_{\mathfrak{i}}^{A^{\mathrm{eq}}}, \mathbf{F}^B, \mathbf{R}^B)$ for $\mathbf{F}$ and $\mathbf{R}$ function and relation symbols of L, with $\mathbf{F}^B$ the restriction of $\mathbf{F}^{A^{\mathrm{eq}}}$ to the* *appropriate power of B and $\mathbf{R}^B$ the intersection of $\mathbf{R}^{A^{\text{eq}}}$ with the appropriate power of B, then $f_{\mathfrak{i}}$ is an isomorphism from A onto B.* $\square$

Now let

$$E = \bigcup_{n\in\omega} (\{n\} \times \mathscr{P}((A^{\text{eq}})^n)).$$

For any $F \subseteq E$ and $n \in \omega\backslash 1$ let $F^{(n)} = \{R \subseteq (A^{\text{eq}})^n : (n, R) \in F\}$.

For positive integers m, n, a subset f of $(A^{\text{eq}})^{m+n}$ is an (m, n)-*function* iff for all $\overline{a} \in (A^{\text{eq}})^m$ there is at most one $\overline{b} \in (A^{\text{eq}})^n$ such that $\overline{a}{}^\frown\overline{b} \in f$. We let $\hat{f}$ be the associated actual function from a subset of $(A^{\text{eq}})^m$ into $(A^{\text{eq}})^n$. Conversely, if g is a function from $X \subseteq (A^{\text{eq}})^m$ into $(A^{\text{eq}})^n$, we define

$$\check{g} = \{\overline{a}{}^\frown\overline{b} : \overline{a} \in X \text{ and } \overline{b} = g(\overline{a})\}.$$

Thus $\check{g}$ is then a (m, n)-function. Note that $\hat{}$ and $\check{}$ are inverses of each other.

For m, n positive integers, $\pi_1^{m,n}$ is the $(m + n, m)$-function defined by

$$\check{\pi}_1^{m,n}(\overline{a}) = \overline{a} \restriction m.$$

Similarly, $\pi_2^{m,n}$ is the $(m + n, n)$-function defined by

$$\check{\pi}_2^{m,n}(\overline{a}) = \langle a_m, \dots, a_{m+n-1}\rangle.$$

A *universe over A* is a subset F of E with the following properties:

(1) For each positive integer m, $(A^{\text{eq}})^m \in F^{(m)}$, and $F^{(m)}$ is closed under union, intersection, and complementation with respect to $(A^{\text{eq}})^m$.

(2) (a) If $R \in F^{(m)}$ and $S \in F^{(n)}$, then $\{\overline{a}{}^\frown\overline{b} : \overline{a} \in R \text{ and } \overline{b} \in S\} \in F^{(m+n)}$.
(b) If m and n are positive integers, then $\pi_1^{m,n} \in F^{(2m+n)}$ and $\pi_2^{m,n} \in F^{(2n+m)}$.
(c) If m and n are positive integers and $R \in F^{(m+n)}$, then $\hat{\pi}_1^{m,n}[R] \in F^{(m)}$ and $\hat{\pi}_2^{m,n}[R] \in F^{(n)}$.
(d) $\Delta_2 \overset{\text{def}}{=} \{(a, a) : a \in A^{\text{eq}}\} \in F^{(2)}$.

(3) If m is a positive integer and $v \in R \in F^{(m)}$, then $\{v\} \in F^{(m)}$.

(4) If $e \in \text{Eq}$, then $\{(\overline{a}/e, e) : \overline{a} \in \partial_e\} \in F^{(1)}$ and $\{(f_{\mathfrak{i}} \circ \overline{a})^\frown\langle(\overline{a}/e, e)\rangle : \overline{a} \in \partial_e\} \in F^{(\text{len}(e)+1)}$.

Lemma 2. *If F is a universe over A and $f \in F^{(m+n)}$ is an (m, n)-function, then* $\text{dmn}(\hat{f}) \in F^{(m)}$ *and* $\text{rng}(\hat{f}) \in F^{(n)}$.

Proof. Note that $\hat{\pi}_1^{m,n}[f] \in F^{(m)}$, and

$$\begin{aligned}\hat{\pi}_1^{m,n}[f] &= \{\overline{b} : \exists\overline{a} \in f\, [\hat{\pi}_1^{m,n}(\overline{a}) = \overline{b}]\} \\ &= \{\overline{b} : \exists\overline{a} \in f\, [\overline{a} \restriction m = \overline{b}]\} \\ &= \text{dmn}(\hat{f}).\end{aligned}$$

Also, we have $\hat{\pi}_2^{m,n}[f] \in F^{(n)}$, and

$$\begin{aligned}\hat{\pi}_2^{m,n}[f] &= \left\{\overline{b} : \exists \overline{a} \in f\left[\hat{\pi}_2^{m,n}(\overline{a}) = \overline{b}\right]\right\} \\ &= \left\{\overline{b} : \exists \overline{a} \in f\left[\langle a_m, \ldots, a_{m+n-1}\rangle = \overline{b}\right]\right\} \\ &= \operatorname{rng}(f).\end{aligned}$$

□

For each positive integer n, let $\Delta_n = \{(\underbrace{a, a, \ldots, a}_{n\ a's}) : a \in A^{\mathrm{eq}}\}$.

Lemma 3. *Let A be an L-structure. For each positive integer m let*

$$B^{(m)} = \{\{\overline{a} \in (A^{\mathrm{eq}})^m : A^{\mathrm{eq}} \models \varphi[\overline{a}]\} : \varphi \textit{ is a formula with free variables among } v_0, \ldots, v_{m-1}\}.$$

Then $F \stackrel{\mathrm{def}}{=} \bigcup_{m=1}^{\infty}(\{m\} \times B^{(m)})$ is a universe.

Proof. (1) is clear. For (2)(a), let $R = \{\overline{a} \in (A^{\mathrm{eq}})^m : A^{\mathrm{eq}} \models \varphi[\overline{a}]\}$ and $S = \{\overline{b} \in (A^{\mathrm{eq}})^n : A^{\mathrm{eq}} \models \psi[\overline{b}]\}$. With $\overline{x}$ and $\overline{y}$ disjoint sequences of distinct variables of lengths m, n respectively, clear $\varphi(\overline{x}) \wedge \psi(\overline{y})$ is the formula needed for (2)(a).

For (2)(b), the following formulas work, where $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ are pairwise disjoint sequences of distinct variables of lengths m, n, m, n respectively:

$$\bigwedge_{i<m} x_i = z_i;$$

$$\bigwedge_{i<n} y_i = w_i.$$

The first formula is considered as $\varphi(\overline{x}, \overline{y}, \overline{z})$, defining $\pi_1^{m,n}$, and the second formula is considered as $\psi(\overline{x}, \overline{y}, \overline{w})$, defining $\pi_2^{m,n}$.

For (2)(c), let R be defined by $\varphi(\overline{x}, \overline{y})$. Then the following formulas define $\hat{\pi}_1^{m,n}[R] \in F^{(m)}$ and $\hat{\pi}_2^{m,n}[R] \in F^{(n)}$ respectively: $\exists \overline{y} \varphi(\overline{x}, \overline{y})$ and $\exists \overline{x} \varphi(\overline{x}, \overline{y})$.

Δ_2 is defined by $x = y$.

$\{v\}$ is defined by $\bigwedge_{i<m} x_i = v_i$ (a definition with parameters).

$\{(\overline{a}/e, e) : \overline{a} \in \partial_e\}$ is defined by $P_e(x)$.

Finally, $\{(f_{\mathfrak{i}} \circ \overline{a})^\frown \langle(\overline{a}/e, e)\rangle : \overline{a} \in \partial_e\}$ is defined by

$$\bigwedge_{i<m} P_{\mathfrak{i}}(x_i) \wedge x_m = \mathscr{F}_e(\overline{x}).$$

□

Since a universe has been defined as a subset F of E containing certain sets (like $(1, A^{\mathrm{eq}})$ and Δ_2), and closed under certain partial operations (like $(n, R) \mapsto (n, (A^{\mathrm{eq}})^m)$ for $R \subseteq (A^{\mathrm{eq}})^n$ and $(m+n, R) \mapsto (m, \hat{\pi}_1^{m,n}[R])$ for $R \subseteq (A^{\mathrm{eq}})^{m+n}$), it follows that any subset F of E is contained in a unique unverse over A.

Lemma 4. *If F is a universe, then $\Delta_n \in F^{(n)}$ for every positive integer n.*

Proof. For $n = 1$ this is true since $A^{\mathrm{eq}} \in F^{(1)}$. For $n = 2$ it is true by (2)(d). Now assume that $n \geq 2$ and it is true for n. Then applying (2)(a) to Δ_n and A^{eq} we get $E \stackrel{\mathrm{def}}{=} \{(\underbrace{a, a, \ldots, a}_{n\ a's}, b) : a, b \in A^{\mathrm{eq}}\} \in F^{(n+1)}$. Now $\pi_1^{1,n-1} = \{(a_0, a_1, \ldots, a_{n-1}, a_0) : a_0, \ldots, a_{n-1} \in A^{\mathrm{eq}}\}$, so $\Delta_{n+1} = E \cap \pi_1^{1,n-1} \in F^{(n+1)}$. $\square$

Lemma 5. *If F is a universe, f is an (m,n)-function in $F^{(m+n)}$, and $M \in F^{(m)}$, then*

$$f \restriction M \stackrel{\mathrm{def}}{=} \{\overline{a} \in f : \overline{a} \restriction m \in M\}$$

is an (m,n)-function and is also in $F^{(m+n)}$.

Proof. By (1) and (2)(a), $N \stackrel{\mathrm{def}}{=} \{\overline{a} \in (A^{\mathrm{eq}})^{(m+n)} : \overline{a} \restriction m \in M\}$ is in $F^{(m+n)}$. Clearly $f \restriction M = f \cap N$. $\square$

Lemma 6. *If F is a universe, f is an (m,n)-function in $F^{(m+n)}$, and $\emptyset \neq M \in F^{(m)}$ with $M \subseteq \mathrm{dmn}(f)$, then $\hat{f}[M] \in F^{(n)}$.*

Proof. By Lemmas 2 and 5. $\square$

Lemma 7. *If F is a universe, f is an (m,n)-function, and $P \in F^{(n)}$, then $\hat{f}^{-1}[P] \in F^{(m)}$.*

Proof. By (1) and (2)(a), $Q \stackrel{\mathrm{def}}{=} \{\overline{a} \in (A^{\mathrm{eq}})^{(m+n)} : \langle a_m, \ldots, a_{m+n-1}\rangle \in P\}$ is in $F^{(m+n)}$. Clearly $\hat{f}^{-1}[P] \in F^{(m)} = \hat{\pi}_1^{m,n}[f \cap Q]$. $\square$

Now for each positive integer m and each $i < m$ let $\pi_3^{m,i} : (A^{\mathrm{eq}})^m \to A^{\mathrm{eq}}$ be defined by

$$\hat{\pi}_3^{m,i}(a_0, \ldots, a_{m-1}) = a_i.$$

Lemma 8. *For F a universe, $\pi_3^{m,i} \in F^{(m+1)}$.*

Proof. For $i = 0$ and $m = 1$, $\pi_3^{m,i} = \Delta_2$. For $i = 0$ and $m > 1$, $\pi_3^{m,i} = \pi_1^{1,m-1}$. For $m > 1$ and $i = m-1$, by (1) and (2)(a),(d) applied to $(A^{\mathrm{eq}})^{m-1}$ and Δ_2, we have $\pi_3^{m,i} = \{\langle a_0, \ldots, a_{m-1}, a_{m-1}\rangle : \overline{a} \in (A^{\mathrm{eq}})^m\} \in F^{(m+1)}$. Finally, for $m > 2$ and $0 < i < m-1$, by (1) and (2)(a) applied to $(A^{\mathrm{eq}})^i$ and $\pi_3^{m-i,0}$ we have

$$\pi_3^{m,i} = \{\langle a_0, \ldots, a_{i-1}, a_i, \ldots, a_{m-1}, a_i\rangle : \overline{a} \in (A^{\mathrm{eq}})^m\} \in F^{(m+1)}. \qquad \square$$

Now suppose that m is a positive integer and $\emptyset \neq M \subseteq m$. Write $M = \{i_0, \ldots, i_{n-1}\}$ with $i_0 < \cdots < i_{n-1}$. Then we define $\pi_4^{m,M}$ to be the (m,n)-function such that

$$\hat{\pi}_4^{m,M}(\overline{a}) = \langle a_{i_0}, \ldots, a_{i_{n-1}}\rangle.$$

Lemma 9. *If F is a universe over A, then $\pi_4^{m,M} \in F^{(m+n)}$.*

Proof. We prove this by induction on n. For $n = 1$ it is true by Lemma 8. Now assume it for n, and suppose that $M \subseteq m$ with $|M| = n+1$, say $M = \{i_0, \ldots, i_n\}$ with $i_0 < \cdots < i_n$. Let $N = \{i_0, \ldots, i_{n-1}\}$. So $\pi_4^{m,N} \in F^{(m+n)}$ by the inductive hypothesis.

Then $\hat{\pi}_3^{m+n,i_n}[\pi_4^{m,N}] \in F^{(m+n+1)}$ by Lemma 6, and for $\overline{a} \in (A^{\mathrm{eq}})^m$, $\overline{b} \in (A^{\mathrm{eq}})^n$, and $c \in A^{\mathrm{eq}}$,

$$\begin{aligned} \overline{a}^\frown \overline{b}^\frown \langle c \rangle \in \hat{\pi}_3^{m+n,i_n}[\pi_4^{m,N}] \quad & \text{iff} \quad \overline{a}^\frown \overline{b} \in \pi_4^{m,N} \text{ and } c = a_{i_n} \\ & \text{iff} \quad \overline{b} = \langle a_{i_0}, \ldots, a_{i_{n-1}} \rangle \text{ and } c = a_{i_n} \\ & \text{iff} \quad \overline{b}^\frown \langle c \rangle = \langle a_{i_0}, \ldots, a_{i_n} \rangle; \end{aligned}$$

Thus $\hat{\pi}_3^{m+n,i_n}[\pi_4^{m,N}] = \pi_4^{m,M}$, as desired. $\square$