The Paris-Harrington Theorem:

An Introduction to "Natural" Incompleteness

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  No reference to classical or mainstream mathematical objects.
- Motivates a search for independent sentences of a more "natural" flavor.

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- Φ actually *equivalent* to 1-consistency of PA and transfinite induction through ϵ_0.

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- A finite ("miniature") version is provable using König's lemma on finitely branching, infinite trees:

$$\forall l, s, e, c \exists u \Big([l, u] \longrightarrow (s)^e_c \Big)$$

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Strategy suggests looking for Φ which are variants of Ramsey's Theorem.

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Remark. Notions of largeness for finite sets of integers play prominent role in modern independence proofs.

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- Without the $*$ underneath the arrow, this is just the Finite Ramsey Theorem (which is provable in PA).
- PH is a consequence of the Infinite Ramsey Theorem and the proof *cannot* be carried out in PA.

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Proof. Fix e, c, s and suppose otherwise that no such interval $[0, u]$ exists satisfying PH. Call P a counterexample for $[0, u]$ if P partitions $[0, u]^e$ into c colors but no $H \subset [0, u]$ which is homogeneous for P is also relatively large.

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Organize the collection of counterexamples into an infinite, finitely branching tree T. For P and P' counterexamples, put $P <_T P'$ if $u < u'$ and P is the restriction of P' to $[0, u]^e$.

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By IRT there is an infinite $H \subseteq \omega$ homogeneous for P. But by choosing u large enough compared to s and $\min(H)$, we have $H \cap [0, u]$ is relatively large and homogeneous for P restricted to $[0, u]^e$, a contradiction. $\diamondsuit$

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- *Remark.* The proof indicates many "self-refining" variants of PH, i.e., there are homogeneous sets H with $\min(H) + s < |H|$, $(\min H)^2 < |H|$, etc.

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- PA proves $Con(T) \rightarrow Con(PA)$.
- *Remark.* Nowadays, combinatorics of original proof is bypassed by showing PH implies KM (another combinatoric principle) and then showing independence of KM from PA.

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Lemma. Given $P_1 : [0, u]^e \to c_1$ and $P_2 : [0, u]^e \to c_2$, there is $P : [0, u]^e \to c_1 c_2$ where any $H \subseteq [0, u]$ is homogeneous for P iff also homogeneous for both P_1 and P_2.

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Lemma. Given $P : [0,u]^e \to c$ there is $P' : [0,u]^{e+1} \to (1 + 2\sqrt{c})$ where any $H \subseteq [0,u]$ of size $> e+1$ is homogeneous for P iff also homogeneous for P'.

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- *Corollary:* Relatively large homogeneous sets can be sufficiently “spread out". (This is what we want!)

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Lemma. ♣: For all e, s, c there is a u such that for any family $\{P_\alpha : \alpha < 2^u\}$ of partitions $P_\alpha : [0, u]^e \to c$ there is an X of size at least s such that

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Lemma. PH implies ♣.

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Given a formula $\theta(\bar{y})$ in prenex form (in the language of PA), say $\theta(\bar{y}) = \exists x_1 \ldots \forall x_r \varphi(\bar{x}, \bar{y})$ define another formula $\theta^*(\bar{y}; \bar{z}) = \exists x_1 < z_1 \ldots \forall x_r < z_r \varphi(\bar{x}, \bar{y}; \bar{z})$.

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Remark. Above proof can be formalized and carried out in PA.

Model-Theoretic Proof of PH

Theorem:(Bovykin) The statement "for all e, s, c there exists u such that for every $P : [0, u]^e \to c$ there is $H \subseteq [0, u]$ homogeneous for P with $\max(s, e \cdot (2^{e \cdot \min(H)} + 1)) < |H|$" is not a theorem of PA.

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Proof. WLOG, let $c = 2$, $\mathcal{M} \models PA$ be non-standard, pick d a non-standard integer. Let $\varphi_1(x_1, \ldots, x_d, y), \ldots, \varphi_d(x_1, \ldots, x_d, y)$ be an enumeration of the first d Δ_0 formulas in at most the free variables shown.

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Let u be the minimal point in $\mathcal{M}$ such that for every $P : [0, u]^{2d+1} \rightarrow 2$ there is a P-homogeneous $H \subseteq [0, u]$ with $|H| > d \cdot (2^{d \cdot \min(H)} + 1)$.

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Note that exponent e in this case is really $2d + 1$.

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Using the definition of u, extract a P'-homogeneous set $H \subseteq [0, u]$ of size greater than $d \cdot (2^{d \cdot \min(H)} + 1)$.

Model-Theoretic Proof of PH

Proof. cont'd. For every increasing d-tuple $b_1 < \cdots < b_d$ in $H - \{\min(H)\}$ define the following sequence of d-many subsets of $[0, \min(H))$:

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Thank You