Nilpotent Groups of Finite Morley Rank

The main reference for the following is *Groups of Finite Morley Rank* by A. Borovik and A. Nesin. *Simple Groups of Finite Morley Rank*, a book in preparation by T. Altinel, A. Borovik, and G. Cherlin, refines some of the ideas in *Groups of Finite Morley Rank* and serves as a reference for relative definability.

1 Properties of Nilpotent Groups of FMR

We begin with a simple lemma.

Lemma 1. *Let G be a group of FMR and H be a connected subgroup of G. For all $g \in G$, if $[g, H]$ is finite, then in fact $[g, H] = 1$.*

Proof. The the fibers of the commutator map $[g, -] : H \longrightarrow [g, H]$ are the right cosets of $C_H(g)$ in H, so $C_H(g)$ has finite index in H. Noting that $C_H(g) = H \cap C_G(g)$ is relatively definable in H, the connectedness of H implies that $C_H(g) = H$. $\square$

Theorem 2. *Let G be a nilpotent group of FMR and H a subgroup of G.*

(a) If H is infinite and G-normal, then $H \cap Z(G)$ is infinite.

(b) If $|G : H|$ is infinite and H is definable, then $|N_G(H) : H|$ is infinite.

Proof. First assume that H is infinite and G-normal. Choose i minimal such that $A := H \cap Z_{i+1}(G)$ is infinite (here we are using that G is nilpotent). Then A° is also infinite. Additionally, A° is a connected subgroup of G such that $[G, A^\circ] \leq [G, A] \leq H \cap Z_i(G)$. The latter is finite, so by Lemma 1, $[G, A^\circ] = 1$. Thus A° is an infinite subgroup of H, central in G, so we have proved part (a).

Now assume $|G : H|$ is infinite and that H is definable in G (but H need no longer be infinite or G-normal). Let $Z = Z^\circ(G)$, and note that (a) implies that Z is infinite. If $|Z : H \cap Z|$ is infinite, then $|HZ : H|$ is infinite as well. In this case we are done since $H \leq HZ \leq N_G(H)$. Otherwise, $|Z : H \cap Z|$ is finite, so the connectedness of Z implies that $Z \leq H$. We can now proceed by induction on the rank of G to get that $|N_{G/Z}(H/Z) : H/Z|$ is infinite. Now $N_{G/Z}(H/Z)/(H/Z) = (N_G(H)/Z)/(H/Z) \cong N_G(H)/H$.

The last statement follows from the fact that if K is the unique subgroup containing Z such that $K/Z = N_{G/Z}(H/Z)$ then for $k \in K$, $\overline{H}^{\overline{k}} = \overline{H}$ implies that $H^k = HZ = H$ (where the bar denotes passage to the quotient G/Z). This tells us that $K \subseteq N_G(H)$. The reverse inclusion is clear. $\square$

We now derive other consequences of Lemma 1 including a description of the minimal, infinite, definable subgroups of a group of FMR. We are working to prove the following theorem.

Theorem 3. (Reineke, 1975) *In a group of FMR, a minimal, infinite, definable subgroup A is abelian. In fact, A is divisible or an elementary abelian p-group.*

To prove the second sentence of the theorem, we will need to know about abelian groups of FMR.

Theorem 4. (Macintyre, 1971) *Let G be an abelian group of FMR. Then $G = DC$ where $D = T \times N$ and*

D is definable, characteristic, divisible, and connected.

C is definable, characteristic, and of bounded exponent.

T is the torsion part of D and is divisible.

N is torsion free and divisible.

Further, $D \cap C$ is finite, and if G is connected, we can take C to be connected.

Proof. Let $G(n) = \{g^n : g \in G\}$, and set $D = \bigcap_{n\in\mathbb{N}} G(n) = \bigcap_{n\in\mathbb{N}} G(n!)$. Then D is a divisible subgroup of G, and by DCC, $D = G(n!)$ for some n. By a theorem of Baer, stating that in the category of abelian groups divisible groups are injective, D has a complement in G, call it B. Considering G/D, we see that B has bounded exponent of at most $n!$. Set $C = \{g \in G : g^{\exp(B)} = 1\}$. Because $B \leq C$, $G = DC$. Further, D and C are characteristic because they are 0-definable, and divisible groups are connected (they can have no nontrivial, finite quotients and the presence of any nontrivial subgroup of finite index implies the presence of a nontrivial, normal subgroup of finite index).

Setting T to be the torsion part of D, it is easily checked that T is divisible. Again by Baer, T has a complement in D, call it N, which must be torsion free. As N is a quotient of D, it is also divisible.

Finally, we show that $D \cap C$ is finite. It will then follow that if G is connected, we have $G = DC^\circ$ (using that $\mathrm{rk}\, G = \mathrm{rk}\, D + \mathrm{rk}\, C^\circ - \mathrm{rk}\,(D \cap C^\circ)$). We now show that for $k \in \mathbb{N}$, D has only finitely many elements of order dividing k (our proof works for any k-divisible, abelian group of FMR). Set $D_{k^m} = \{x \in D : x^{k^m} = 1\}$. We want to show D_k is finite. If D_k is trivial, we are done. Otherwise, let $y_1 \in D_k \setminus \{1\}$. By the divisibility of D, there exists a $y_2 \in D$ such that $y_2^k = y_1$, so $y_2 \in D_{k^2} \setminus D_k$. Repeating, we see that (D_{k^m}) is a strictly increasing sequence of definable subgroups. Further, $D_{k^{m+1}}/D_{k^m}$ is isomorphic to D_k via the interpretable map $xD_{k^m} \mapsto x^{k^m}$. If D_k is not finite, $(\mathrm{rk}(D_{k^m}))$ is a strictly increasing sequence, which is a contradiction. □

We need two more short lemmas before we reach our goal.

Lemma 5. *If G is a group of FMR and H a connected subgroup of G such that $Z(H)$ is finite, then $H/Z(H)$ is a centerless group.*

Proof. We wish to show $Z_2(H) = Z(H)$. Let $w \in Z_2(H)$, so that $[w, H] \leq Z(H)$. By Lemma 1, $[w, H] = 1$, so w is central. □

Lemma 6. *If G is a connected group of FMR such that $C_G(x)$ is finite for all $x \in G \setminus \{1\}$, then $G = 1$.*

Proof. If G is finite, the connectedness of G implies that $G = 1$. By way of contradiction, assume that G is infinite. We know that G is the disjoint union of $\{1\}$ and the nontrivial conjugacy classes of G. However, for all $x \in G \setminus \{1\}$, the right coset space $G/C_G(x)$ is in interpretable bijection with x^G. Thus $\mathrm{rk}\,(x^G) = \mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(x)) = \mathrm{rk}\,(G)$. Since G is degree 1, G must have only one nontrivial conjugacy class. Hence, for any $x \in G \setminus \{1\}$, $G = x^G \cup \{1\}$. Further, $x \in C_G(x)$ which is finite, so x has finite order. Thus every nontrivial element of G has the same finite order, so $\exp(G) = p$ for some prime p.

Now, $N_G(\langle x \rangle)/C_G(x)$ is a finite group with order dividing $|\mathrm{Aut}(\langle x \rangle)| = p-1$, but it certainly must also have exponent dividing p (so p divides the order of $N_G(\langle x \rangle)/C_G(x)$). We conclude that $N_G(\langle x \rangle)/C_G(x)$ is trivial, so $N_G(\langle x \rangle)$ acts trivially on $\langle x \rangle$ (this also follows from the fact that the only definable action of a connected group of FMR on a finite set is the trivial action). If $\exp(G) > 2$, we contradict the fact that $x^2 \in x^G$. If $\exp(G) = 2$, G is abelian, and we contradict the fact that $C_G(x)$ is finite. $\square$

We are now able to prove the theorem of Reineke that we have been working towards.

Proof of Theorem 3. We wish to show that $Z(A) = A$. By the minimality of A, it is enough to show that $Z(A)$ is infinite. Note that A is connected. Towards a contradiction, assume $Z(A)$ is finite. By Lemma 5, $\overline{A} := A/Z(A)$ is centerless, and (by assumption) $\overline{A}$ has no proper infinite, definable subgroups. Thus for all $\overline{a} \in \overline{A}$, we have that $C_{\overline{A}}(\overline{a})$ is finite. Since $\overline{A}$ is connected, this forces $\overline{A}$ to be trival, contradicting the fact that A is infinite. Thus A is abelian.

By the Macintyre's theorem, $A = D * C$ where D is divisible, C has bounded exponent, and both are definable (in A hence in G). Certainly D or C must be infinite, so by the minimality of A the one that is infinite must equal A. If $A = D$, we are done. Otherwise, A has bounded exponent, say $\exp(A) = n$. Let p be a prime dividing n. Then $\varphi : A \longrightarrow A : a \mapsto a^p$ is an endomorphism whose image has exponent n/p, so the image is a proper definable subgroup (of A hence of G). By the minimality of A, the image of φ must be finite, so the kernel of φ is infinite (hence equal to A). $\square$

2 The Structure of Nilpotent Groups of FMR

Our goal is to prove the following theorem which will come in two pieces.

Theorem 7. (Nesin, 1991) *Let G be a nilpotent group of FMR. Then $G = D * C$ where $D = T \times N$ and*

D is definable, characteristic, divisible, and connected.

C is definable, characteristic, and of bounded exponent.

T is the torsion part of D and is divisible and central in G.

N is torsion free.

Further, $D \cap C$ is central and finite, and if G is connected, we can take C to be connected.

We begin by showing that we can decompose G as a central product of a divisible subgroup and a subgroup of bounded exponent. We will later address the decomposition for D.

Lemma 8. *Let G be a nilpotent group of FMR. Then $G = D * C$ where*

D is definable, characteristic, divisible, and connected.

C is definable, characteristic, and of bounded exponent.

Further, $D \cap C$ is central and finite, and if G is connected, we can take C to be connected.

Proof. We begin by explaining why it is enough to show that $G = DB$ for D a divisible subgroup and B a subgroup of finite exponent. Note that D centralizes B (see background). Let $n = \exp(B)$. For $g \in G$, we may write $g = db$ for $d \in D$ and $b \in B$, and $g^n = d^n b^n = d^n$ (since D centralizes B). Thus, $D = \{d^n : d \in D\} = \{g^n : g \in G\}$, so D is in fact 0-definable and characteristic. Now set $C = \{g \in G : g^n = 1\}$. It is not clear that C is a subgroup, but C is 0-definable, hence characteristic, set containing B. To show C is a subgroup we need only show that C is closed under multiplication (inversion and 1 are clear). Using our previous observations $C = \{db \in G : d \in D, b \in B, d^n = 1\} \subseteq \{db \in G : d \in D, b \in B, d \text{ is central in } G\}$ (see background). Thus for $c_1, c_2 \in C$, $(c_1c_2)^n = d_{c_1}^n d_{c_2}^n (b_{c_1} b_{c_2})^n = 1$, so C is a subgroup. Clearly $G = DC$ where D centralizes C, so $G = D * C$. Finally, $D \cap C$ is contained in $\operatorname{Tor}(D)$ which we have already mentioned is central in G. $\operatorname{Tor}(D)$ is easily seen to be divisible, so the argument in Theorem 4 (Macintyre's Theorem) shows that $\operatorname{Tor}(D)$ has only finitely many elements of order dividing n. Hence, $D \cap C$ is finite. Thus, it suffices to show that $G = DB$ for D a divisible subgroup and B a subgroup of finite exponent.

Now suppose that the lemma is not true, and let G be a counterexample of minimal rank and degree. We will show that in fact $G = DB$ for D a divisible subgroup and B a subgroup of finite exponent, which will be our contradiction. Now, G is not abelian (by Macintyre's Theorem). Set $Z = Z^\circ(G)$, and (by minimality) write $Z = D_0C_0$ with D_0 and C_0 as in the lemma with both connected. Note that D_0 and C_0 are central subgroups of G and are actually characteristic in G. We now consider two cases.

Case 1: Assume that $C_0 \neq 1$. Thus, the connectedness of C_0 implies that C_0 is infinite, so $\operatorname{rk}(G/C_0) < \operatorname{rk} G$. By induction, $G/C_0 = (D_1/C_0)*(C_1/C_0)$ where (D_1/C_0) and (C_1/C_0) are as expected. Note that, as C_0 is central, C_1 is of bounded exponent and we have the following lattice.

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
D_1 \arrow[dr, dash] & & C_1 \arrow[dl, dash] \\
 & D_1 \cap C_1 \arrow[d, dash] & \\
 & C_0 &
\end{tikzcd}$$

If $D_1 \neq G$, then we may write $D_1 = D_2 * C_2$ for D_2 divisible and C_2 of bounded exponent. Then $G = D_1C_1 = D_2(C_2C_1)$. Since C_2C_1 is nilpotent and each factor is of bounded exponent, C_2C_1 is of bounded exponent (see background), so we are done with $D = D_2$ and $B = C_2C_1$.

Next consider when $D_1 = G$. First note that G/C_0 is divisible. We now work to produce a proper, definable subgroup H of G such that $G = HC_0 = H*C_0$ (noting that C_0 is central). For then, by minimality, $H = D_3 * C_3$, and we are done with $D = D_3$ and $B = C_3C_0$. Let $n = \exp(C_0)$, and set $X = \{g^n : g \in G\}$ and $H = \langle X \rangle$. Since G/C_0 is n-divisible, $G = XC_0$. We show X is n-divisible. For $x = g^n$ in X, write $g = yc$ for $y \in X$ and $c \in C_0$. Since C_0 is central, $x = g^n = y^n c^n = y^n$, so X is n-divisible. Now, H is nilpotent and H/H' is n-divisible, as it is an abelian group generated by the n divisible set XH', so H is n-divisible (see background). Thus $H = X$ is a 0-definable, characteristic subgroup. As before, $H \cap C_0$ is contained in $\operatorname{Tor}(H)$ which is central in G. $\operatorname{Tor}(H)$ is easily seen to be n-divisible, so $\operatorname{Tor}(H)$ has only finitely many elements of order dividing n. Hence, $H \cap C_0$ is finite. A rank argument shows that $\operatorname{rk}(H) < \operatorname{rk}(G)$, so H is proper. This finishes case 1.

Case 2: Assume that $C_0 = 1$. Then D_0 is infinite, so $\operatorname{rk}(G/D_0) < \operatorname{rk} G$. By induction, $G/D_0 = (D_1/D_0) * (C_1/D_0)$ where (D_1/D_0) and (C_1/D_0) are as expected. Note that, as D_0 is central, D_1 is divisible and we have the previous lattice with D_0 replacing C_0.

If $C_1 \neq G$, then we may write $C_1 = D_2 * C_2$ for D_2 divisible and C_2 of bounded exponent. Then $G = D_1C_1 = (D_1D_2)C_1$. Since D_1D_2 is nilpotent and each factor is divisible, D_1D_2 is divisible (see background), so we are done with $D = D_1D_2$ and $B = C_1$.

Next consider when $C_1 = G$. This time G/D_0 is of bounded exponent. Let $n = \exp(G/D_0)$, and set $X = \{g \in G : g^n = 1\}$ and $B = \langle X \rangle$. We show $G = BD_0$. Let $g \in G$. Then $g^n \in D_0$, so the divisibility of D_0 implies that there is a $d \in D_0$ such that $g^n = d^n$. Because D_0 is central, $(gd^{-1})^n = 1$, so $gd^{-1} \in B$. Thus, $g = bd$ for some $b \in B$, so $G = BD_0$. Now, B is nilpotent and B/B' is of bounded exponent, as it is an abelian group generated by XB', so B is of bounded exponent (see background). Thus, we are done with $D = D_0$, and this finishes case 2. $\square$

Now we address the decomposition for D in Theorem 7.

Lemma 9. *Let D be a divisible nilpotent group of FMR, and $T = Tor(D)$. Then T is a divisible central subgroup of D, and $D = T \times N$ for some torsion free divisible subgroup N.*

Proof. See *Groups of Finite Morley Rank* by A. Borovik and A. Nesin. $\square$