Background Group Theory

Let G be a group, $H \leq G$, and $g \in G$.

1. The commutator map $[g, -] : H \longrightarrow G$

(a) The fibers of $[g, -]$ are the right cosets of $C_H(g)$ in H.

(b) If $[g, H]$ is central, then $[g, -]$ is a homomorphism such that for all $n \in \mathbb{Z}$ and $h \in H$,
$[g, h]^n = [g, h^n] = [g^n, h]$. This will always be true for $g \in Z_2(G)$ or $H \leq Z_2(G)$.

(c) The image of $Z_{i+1}(G)$ under $[g, -]$ is in $Z_i(G)$.

2. Definition: G is n-divisible (or n-radicable) if for all $g \in G$ there is a $x \in G$ such that $x^n = g$. G is divisible if it is n-divisible for all integers $n > 0$.

3. If G is a nilpotent group then any n-divisible subgroup centralizes all elements G of order dividing n. In particular, the torsion elements in a divisible, nilpotent group form a central divisible subgroup.

Proof. Let D be an n-divisible subgroup of G. If G is abelian, we are done. Let $x \in G$ be an element of order dividing n. By induction, $\overline{D}$ centralizes $\overline{x}$ in $\overline{G} := G/Z(G)$, so $[x, D] \leq Z(G)$. Thus $[x, -] : D \longrightarrow Z(G)$ is a homomorphism into a subgroup of exponent n (because $[x, d]^n = [x^n, d] = 1$). The n-divisibility of D forces the image to be trivial, so D centralizes all n-elements. In particular, if D is divisible then $\mathrm{Tor}(D)$ is central in D, hence a subgroup, and any kth root of an element in $\mathrm{Tor}(D)$ is necessarily in $\mathrm{Tor}(D)$. $\square$

4. If G is a nilpotent group then G is n-divisible iff G/G' is n-divisible. In particular, the join of two n-divisible subgroups of a nilpotent group is n-divisible.

Proof. We will show that for $i \geq 1$, G/G^i is n-divisible iff G/G^{i+1} is n-divisible. Certainly the n-divisibility of G/G^{i+1} implies that G/G^i is n-divisible. Now assume that G/G^i is n-divisible. Without loss, we assume that $G^{i+1} = 1$ (and $G \neq G^i$). Then G^i is central, and we will have that G is n-divisible if we can show G^i is n-divisible (here we are using that G/G^i is n-divisible). Now, G^i is an abelian group generated by $\{[g, G] : g \in G^{i-1}\}$. As $[g, G] \subseteq G^i$ is central for each $g \in G^{i-1}$, $[g, -]$ is a homomorphism from G/G^i to G^i. Thus $[g, G]$ is n-divisible for each $g \in G^{i-1}$. Since G^i is abelian, we are done since each generator is contained in an n-divisible subgroup. $\square$

5. If G is a nilpotent group then G is of bounded exponent iff G/G' is of bounded exponent. In particular, the join of two subgroups of bounded exponent in a nilpotent group is of bounded exponent.

Proof. A proof nearly identical to that given for the last item goes through.

We will show that for $i \geq 1$, G/G^i is of bounded exponent iff G/G^{i+1} is of bounded exponent. Certainly if G/G^{i+1} is of bounded exponent then G/G^i of bounded exponent. Now assume that G/G^i is of bounded exponent. Without loss, we assume that $G^{i+1} = 1$ (and $G \neq G^i$). Since groups of bounded exponent are closed under extension, we will have that G is of bounded exponent if we can show G^i is of bounded exponent. Now, G^i is a central (hence abelian) subgroup generated by $\{[g, G] : g \in G^{i-1}\}$. As $[g, G]$ is central for each $g \in G^{i-1}$, $[g, -]$ is a homomorphism from G/G^i to G^i. Thus $[g, G]$ is of bounded exponent for each $g \in G^{i-1}$. Since G^i is abelian, we are done since each generator is contained in a subgroup of bounded exponent. $\square$