Rank. A function $\mathrm{rk}\colon \mathcal{U}\setminus\{\emptyset\}\to\omega$ is a *rank function* if it satisfies axioms (A)–(D) for all $A, B \in \mathcal{U}$.

(A) (Monotonicity of rank) $\mathrm{rk}(A) \geq n+1$ iff there are infinitely many pairwise disjoint, nonempty, $\mathcal{U}$-interpretable subsets of A, each of rank n.

(B) (Definability of rank) If $f\colon A \to B$ is $\mathcal{U}$-interpretable, then for each n the set $\{b \in B \mid \mathrm{rk}(f^{-1}(b)) = n\}$ is $\mathcal{U}$-interpretable.

(C) (Additivity of rank) If $f\colon A \twoheadrightarrow B$ is $\mathcal{U}$-interpretable and if $\mathrm{rk}(f^{-1}(b)) = n$ for all $b \in B$, then $\mathrm{rk}(A) = \mathrm{rk}(B) + n$.

(D) (Elimination of infinite quantifiers) For any $\mathcal{U}$-interpretable $f\colon A \to B$ there is an m such that for any $b \in B$ the set $f^{-1}(b)$ is infinite whenever it contains more than m elements.

Degree. If $A, B \in \mathcal{U}$, then B is a *generic* subset of A if $B \subseteq A$ and $\mathrm{rk}(B) = \mathrm{rk}(A)$. A nonempty set $A \in \mathcal{U}$ has *degree 1* if it is not the disjoint union of two generic $\mathcal{U}$-interpretable sets. A has *degree d* if it is the disjoint union of d generic $\mathcal{U}$-interpretable sets of degree 1.

Lemmata. Let $\mathcal{U}$ be a ranked universe.

(1) An interpretable set has rank 0 iff it is finite and nonempty.

(2) If $A, B \in \mathcal{U}$ and $A \subseteq B$, then $\mathrm{rk}(A) \leq \mathrm{rk}(B)$.

(3) If $B, C \in \mathcal{U}$, then $\mathrm{rk}(B \cup C) = \max(\mathrm{rk}(B), \mathrm{rk}(C))$.

(4) If $A \in \mathcal{U}$, then $\mathrm{rk}(A) \geq n+1$ iff there exist infinitely many subsets A_i of A in $\mathcal{U}$ such that $\mathrm{rk}(A_i) = n$ and $\mathrm{rk}(A_i \cap A_j) < n$ for all $i \neq j$.

(5) The degree of a $\mathcal{U}$-interpretable set is unique if it exists.

(6) If $A, B \in \mathcal{U}$ are disjoint nonempty sets of the same rank and both have degrees, then $A \cup B$ has a degree, and $\deg(A \cup B) = \deg(A) + \deg(B)$.

(7) Every $\mathcal{U}$-interpretable set has a degree.

(8) If $\mathrm{rk}(A) > \mathrm{rk}(B)$, then $\deg(A \cup B) = \deg(A)$.

(9) If $\mathrm{rk}(A) = \mathrm{rk}(B)$ and $A \subseteq B$, then $\deg(A) \leq \deg(B)$.

(10) If $f\colon A \to B$ is a $\mathcal{U}$-interpretable bijection, then A and B have the same rank and degree.

(11) If $A, B \in \mathcal{U}$, then $\mathrm{rk}(A \times B) = \mathrm{rk}(A) + \mathrm{rk}(B)$.

(12) Assume that $A, B, C \in \mathcal{U}$ are nonempty and $C \subseteq A \times B$. For $b \in B$, the sets $A(b) = \{a \in A \mid (a, b) \in C\}$ is in $\mathcal{U}$, and if all have rank r, then $\mathrm{rk}(C) = r + \mathrm{rk}(B)$.

(13) If $A, B \in \mathcal{U}$, then $\deg(A \times B) = \deg(A)\deg(B)$.

(14) If A, B and $f\colon A \to B$ are $\mathcal{U}$-interpretable then $\mathrm{rk}(A) \geq \mathrm{rk}(f(A))$.

(15) If $A \in \mathcal{U}$ is infinite, then there is no relation $<$ in $\mathcal{U}$ that linearly orders A.

Exercise. Show that the ring $\mathbb{R} = \langle \{reals\}; \cdot, +, -, 0, 1\rangle$ does not belong to a ranked universe. Then show that $\mathbb{R}$ is interpretable in the group $\mathrm{GL}_2(\mathbb{R})$ to deduce that this group does not have finite Morley rank.