Solving the Quartic Equation.

Our goal is to learn how to find the roots of $ax^4+bx^3+cx^2+dx+e=0$, where $a \neq 0$. The first person to discover how to do this was Lodovico de Ferrari (who lived 1522-1565), but we will follow Descartes' Method. (Descartes lived 1596-1650, and published his method in 1637).

Stage 1. Divide by leading coefficient and depress the equation. The result will have the form

$$y^4 + py^2 + qy + r = 0.$$

Stage 2. Factor into quadratics. The factorization will look like

$$y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v).$$

Since $(y^2 - sy + u)(y^2 + sy + v) = y^4 + (u + v - s^2)y^2 + (s(u - v))y + uv$, we can equate coefficients in the previous displayed line to obtain a system of polynomial equations

$$\begin{aligned} u + v - s^2 &= p \\ s(u - v) &= q \\ uv &= r \end{aligned}$$

In this system, s, u, v are unknowns while p, q, r are known values from Stage 1. If we solve this system for s, u, v we will have obtained a factorization of the depressed quartic into two quadratic factors.

Let's reduce the system of equations to a single equation in the variable s.

$$\begin{aligned} u + v &= p + s^2 \\ u - v &= q/s, \end{aligned}$$

so $u = \frac{1}{2}(p + s^2 + q/s)$ and $v = \frac{1}{2}(p + s^2 - q/s)$, hence

$$r = uv = \frac{1}{2}\left(p + s^2 + q/s\right) \cdot \frac{1}{2}\left(p + s^2 - q/s\right) = \frac{1}{4}\left((p + s^2)^2 - (q/s)^2\right),$$

or

$$4r = (p^2 + 2ps^2 + s^4) - q^2/s^2,$$

or

$$s^6 + 2ps^4 + (p^2 - 4r)s^2 - q^2 = 0.$$

Now let $S = s^2$. The above equation can be written as

$$S^3 + 2pS^2 + (p^2 - 4r)S - q^2 = 0.$$

This cubic is the *resolvent cubic*.

Using the Cardano Formula, solve $S^3 + 2pS^2 + (p^2 - 4r)S - q^2 = 0$. If the roots are $S = R_1, R_2, R_3$, then $s = \pm\sqrt{R_1}, \pm\sqrt{R_2}, \pm\sqrt{R_3}$. For a given value of s, say $s = \sqrt{R_1}$, we can solve for u and v:

$$\begin{aligned} u &= \tfrac{1}{2}(p + s^2 + q/s) \\ v &= \tfrac{1}{2}(p + s^2 - q/s). \end{aligned}$$

Once we have u, v, s, we have achieved our factorization:

$$y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v).$$

Stage 3. Apply the quadratic formula to each of the quadratics $y^2 - sy + u = 0$, $y^2 + sy + v = 0$ to obtain the roots of the depressed quartic equation $y^4 + py^2 + qy + r = (y^2 - sy + u)(y^2 + sy + v) = 0$. The roots will be

$$\begin{aligned} y_1 &= \tfrac{s+\sqrt{s^2-4u}}{2}, \\ y_2 &= \tfrac{s-\sqrt{s^2-4u}}{2}, \\ y_3 &= \tfrac{-s+\sqrt{s^2-4v}}{2}, \\ y_4 &= \tfrac{-s-\sqrt{s^2-4v}}{2}. \end{aligned}$$

These can be used to solve the original (undepressed) equation.

Practice.

(1) Solve $x^4 - 4x^2 - 8x - 3 = 0$

(2) Solve $x^4 - 10x^2 + 20x - 16 = 0$

(3) Show that if a quartic has the form

$$x^4 + px^2 + r = 0$$

(i.e., there is no qx term in the polynomial), then the Cubic Resolvent is actually a quadratic rather than a cubic. Deduce that quartics of the form of the displayed equation can be solved with no knowledge of the Cubic Formula. Solve the following one:

$$x^4 + 2x^2 + 2 = 0$$