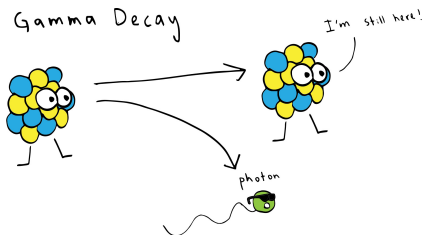


Truth versus Provability



Semantic versus Syntactic Consequence

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We have already discussed how to check whether a statement P is true in a structure (check the tables of the structural elements! play quantifier games!). Today we will discuss provability.

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First question. How is this possible? How do proofs get started?

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So, a “proof system” typically specifies its axioms and also the accepted rules of deduction.

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$$\frac{(\forall x)(P(x) \rightarrow Q(x)), P(s)}{Q(s)}$$

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|-----|-----|-------------------|------------------------------|--|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
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Another way to think about this is: at the first-order level, every statement has a proof or a counterexample.

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Then $\Sigma \models Q$, but $\Sigma \not\vdash Q$ for any proof system requiring finite-length proofs.

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$\neg C, S_2, \dots, \neg H$. \square (Proof by contraposition, or direct proof of the **contrapositive** statement $(\neg C) \rightarrow (\neg H)$.)

Proof structure #3.

$H, \neg C, S_3, \dots, \perp$. \square (Proof by contradiction.)

Examples!

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