A Characterization of Products

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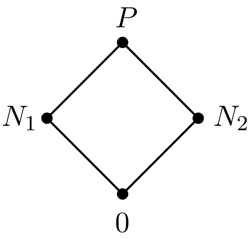

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$$P \overset{\nu_B \times \nu_A}{\longrightarrow} (P/B) \times (P/A) \overset{i \times j}{\cong} A \times B.$$

We have

$$\begin{aligned} (i \times j) \circ (\nu_B \times \nu_A)(ab) &= i(abB) \times j(abA) \\ &= i(aB) \times j(aAb) \end{aligned}$$

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2. The normal subgroups of S_3 form a chain, $\{1\} < A_3 < S_3$. It follows that S_3 cannot be decomposed into a product of smaller groups. (That is, S_3 does not have a pair of proper, nontrivial, complementary, normal subgroups.)