Practice about Pell's equation

Let's find integer solutions to $x^2 - 7y^2 = 1$.

(1) Find a solution with Bhaskara's method starting with the guess $(1, 1)$. Recall that Bhaskara uses Brahmagupta's Identity:

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

- (a) Guess $(1, 1, -6)$
  - (i) Compose with $(m, 1, m^2 - 7)$ to get $(m + 7, m + 1, -6(m^2 - 7))$.
  - (ii) Divide: $((m + 7)/(-6), (m + 1)/(-6), (m^2 - 7)/(-6))$.
  - (iii) Choose $m = 5$, get $(-2, -1, -3)$.
- (b) Guess $(-2, -1, -3)$
  - (i) Compose with $(m, 1, m^2 - 7)$ to get $(-2m - 7, -m - 2, -3(m^2 - 7))$.
  - (ii) Divide: $((-2m - 7)/(-3), (-m - 2)/(-3), (m^2 - 7)/(-3))$.
  - (iii) Choose $m = 1$, get $(3, 1, 2)$.
- (c) Guess $(3, 1, 2)$
  - (i) Compose with $(m, 1, m^2 - 7)$ to get $(3m + 7, m + 3, 2(m^2 - 7))$.
  - (ii) Divide: $((3m + 7)/(2), (m + 3)/(2), (m^2 - 7)/(2))$.
  - (iii) Choose $m = 3$, get $(8, 3, 1)$. ✓

(2) Find a solution using the fact that $\sqrt{7} = [a_0; a_1, a_2, \ldots] = [2; \overline{1, 1, 1, 4}]$. (It may help to remember that the convergents p_n/q_n may be generated by the recurrences $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$.)

$$\left(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right) = \left(\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \boxed{\frac{8}{3}}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \frac{590}{223}, \ldots\right), \ (x, y) = (8, 3).$$

(3) Generate another solution to $x^2 - 7y^2 = 1$.

$$\left(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right) = \left(\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \boxed{\frac{127}{48}}, \frac{590}{223}, \ldots\right), \ (x, y) = (127, 48).$$