The Cattle Problem of Archimedes

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$$\approx 7.76 \times 10^{206544} \text{ cattle.}$$

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7. (1768 CE) First rigorous proof that the continued fractions solution to $x^2 - dy^2 = 1$ will solve the problem due to Joseph-Louis Lagrange.

Brahmagupta's Identity

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2. Composing $(10, 1, 8)$ with itself yields another solution, $(192, 20, 64)$.
3. "Dividing" by 64 yields another $(24, 5/2, 1)$.
4. Composing $(24, 5/2, 1)$ with itself yields $(1151, 120, 1)$.
5. This means that $(x, y) = (1151, 120)$ satisfies $x^2 - 92y^2 = 1$.

Application

Consider Brahmagupta's problem.

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7. The next result obtained this way is $(2649601, 276240, 1)$.