HW 9: solution sketches

(1) Suppose that $f(x)$ is a 1-variable polynomial of degree greater than 1. Explain how to find the points at infinity on the curve $y = f(x)$.

Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n > 1$. Then the homogenization of $y = f(x)$ is $yz^{n-1} = a_n x^n + a_{n-1}x^{n-1}z + \cdots + a_1 x z^{n-1} + a_0 z^n$. The points at infinity are found by setting $z = 0$, which yields $0 = a_n x^n$. From this it follows that $x = 0$. Thus a point at infinity has homogeneous coordinates

$$\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the only point at infinity on $y = f(x)$.

(2) Is there a (complex) polynomial $F(x, y)$ such that the projective completion of the curve defined by $F(x, y) = 0$ has no points at infinity?

No. Bézout's Theorem implies that any line must intersect any curve in the projective plane.

(3) Everybody knows that 2 points are sufficient to determine a line. How many points are sufficient to determine an irreducible conic in the projective plane over $\mathbb{C}$?

Five points determine an irreducible conic in any of the planes $\mathbb{R} \subseteq \mathbb{RP}^2 \subseteq \mathbb{CP}^2$.

Four points are insufficient, in general, since it is easy to find two irreducible conics that intersect in four distinct points — those four intersection points do not determine which of the two conics you are interested in. (For example, it is easy to find two ellipses that intersect in four points.)

But five points are enough, by Bézout's Theorem. If $F(x, y) = 0$ and $G(x, y) = 0$ define distinct degree-2 curves Γ_1 and Γ_2, and these curves have more than $2 \cdot 2 = 4$ points of intersection, then the curves must share a component. The component would have to be a degree-1 curve. Thus, each Γ_i would have to be a union of two lines, and Γ_1 and Γ_2 would have to share a line. But the union of two lines is a reducible curve and this problem is about irreducible curves only, so this case can't happen. Thus, if Γ_1 and Γ_2 are irreducible algebraic curves of degree 2, they share 4 points $\mathbb{CP}^2$.