HW 8: solution sketches

Cardano's Formula expresses the roots of $y^3 - py - q = 0$ as

$$\begin{aligned} y_1 &= \sqrt[3]{\tfrac{q}{2} + \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} + \sqrt[3]{\tfrac{q}{2} - \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} \\ y_2 &= \omega\sqrt[3]{\tfrac{q}{2} + \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} + \omega^2\sqrt[3]{\tfrac{q}{2} - \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} \\ y_3 &= \omega^2\sqrt[3]{\tfrac{q}{2} + \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} + \omega\sqrt[3]{\tfrac{q}{2} - \sqrt{\left(\tfrac{q}{2}\right)^2 - \left(\tfrac{p}{3}\right)^3}} \end{aligned}$$

where $\omega = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

(1) Exercise 6.5.2 from the text: Use Cardano's formula to solve $y^3 - 2 = 0$. Do you get the obvious solution?

$p = 0$ and $q = 2$, so the roots are

$$\begin{aligned} y_1 &= \sqrt[3]{1 + \sqrt{1^2 - 0^3}} + \sqrt[3]{1 - \sqrt{1^2 - 0^3}} = \sqrt[3]{2} + 0 = \sqrt[3]{2} \\ y_2 &= \omega\sqrt[3]{1 + \sqrt{1^2 - 0^3}} + \omega^2\sqrt[3]{1 - \sqrt{1^2 - 0^3}} = \omega\sqrt[3]{2} + \omega^2 0 = \omega\sqrt[3]{2} \\ y_3 &= \omega^2\sqrt[3]{1 + \sqrt{1^2 - 0^3}} + \omega\sqrt[3]{1 - \sqrt{1^2 - 0^3}} = \omega^2\sqrt[3]{2} + \omega 0 = \omega^2\sqrt[3]{2}. \end{aligned}$$

$y_1 = \sqrt[3]{2}$ is the obvious (=real) solution.

(2) Find all of the roots of the sextic $x^6 - 15x^2 - 4 = 0$ using an adapted form of the Cardano formula. (That is, let $y = x^2$, solve $y^3 - 15y - 4 = 0$ with the Cardano formula, then find x.) Which of your roots is equal to the root $x = 2$?

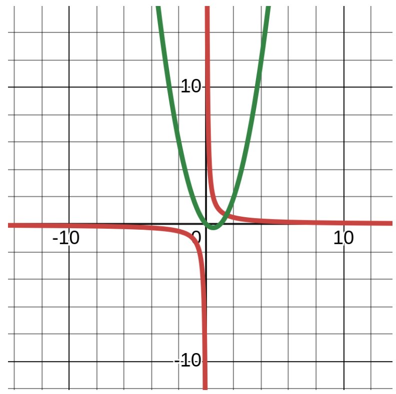
Applying the Cardano Formula to $y^3 - 15y - 4 = 0$ yields

$$\begin{aligned} y_1 &= \sqrt[3]{2 + \sqrt{2^2 - 5^3}} + \sqrt[3]{2 - \sqrt{2^2 - 5^3}} \\ &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= (2 + i) + (2 - i) \\ &= 4 \\ y_2 &= \omega\sqrt[3]{2 + 11i} + \omega^2\sqrt[3]{2 - 11i} \\ &= \omega(2 + i) + \omega^2(2 - i) = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i)(2 + i) + (-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i)(2 - i) \\ &= -2 - \sqrt{3} \\ y_3 &= \omega^2\sqrt[3]{2 + 11i} + \omega\sqrt[3]{2 - 11i} \\ &= \omega^2(2 + i) + \omega(2 - i) \\ &= -2 + \sqrt{3} \end{aligned}$$

The six roots of the original equation are $x = \pm\sqrt{y_1}, \pm\sqrt{y_2}, \pm\sqrt{y_3}$. Only one of these is a positive real, namely $+\sqrt{y_1} = 2$.

(3) Sketch the parabola $y = x^2 - x$ and the hyperbola $xy = 1$ together, and locate all (real) points of intersection of these curves.

According to the graph below there should be one real point of intersection.



To find it, solve the system

$$\begin{aligned} y &= x^2 - x \\ xy &= 1. \end{aligned}$$

The goal is to find the coordinates (x, y) of the point of intersection. Eliminating y from these equations yields $x^3 - x^2 - 1 = 0$ for the x-value at the point of intersection. The substitution $x = z + \frac{1}{3}$ depresses the equation to $z^3 - \frac{1}{3}z - \frac{29}{27} = 0$. The Cardano formula yields

$$\begin{aligned} z &= \sqrt[3]{\frac{29}{54} + \sqrt{\left(\frac{29}{54}\right)^2 - \left(\frac{1}{9}\right)^3}} + \sqrt[3]{\frac{29}{54} - \sqrt{\left(\frac{29}{54}\right)^2 - \left(\frac{1}{9}\right)^3}} \\ &= \frac{1}{6}\left(\sqrt[3]{116 + 12\sqrt{93}} + \sqrt[3]{116 - 12\sqrt{93}}\right). \end{aligned}$$

Thus $x = \frac{1}{3} + \frac{1}{6}\left(\sqrt[3]{116 + 12\sqrt{93}} + \sqrt[3]{116 - 12\sqrt{93}}\right)$, which is real, and it follows from this that $y = \frac{1}{x} = \frac{1}{\frac{1}{3} + \frac{1}{6}\left(\sqrt[3]{116+12\sqrt{93}} + \sqrt[3]{116-12\sqrt{93}}\right)}$.