

HW 6: solution sketches

- (1) Find the continued fraction for $\frac{34}{21}$ and use it to find a solution to this instance of Bézout's Identity: $21x + 34y = 1$.

$\frac{34}{21} = [1; 1, 1, 1, 1, 1, 1]$, so the convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}.$$

Using the rule $p_{k+1}q_k - p_kq_{k+1} = (-1)^k$ (from Line 2 of Page 5 of the slides on Continued Fractions) we derive that $34 \cdot 13 - 21 \cdot 21 = (-1)^6 = 1$. Thus, $(x, y) = (-21, 13)$ is a solution to $21x + 34y = 1$.

- (2) Use Brahmagupta's method to find a solution to $x^2 - Dy^2 = 1$, where $D = n^2 + 1$.

Choosing $x = n, y = 1$ leads to the starting triple $(x, y, k) = (n, 1, -1)$. Composing it with itself yields a triple $(n^2 + (n^2 + 1)1, 2n, 1) = (2n^2 + 1, 2n, 1)$. Thus $(x, y) = (2n^2 + 1, 2n)$ is a solution.

- (3) The quadratic mean of a sequence a_1, \dots, a_n is

$$\sqrt{\frac{a_1^2 + \dots + a_n^2}{n}}.$$

Find an integer $n > 1$ such that the quadratic mean of the first n positive integers is again an integer. That is, find $n > 1$ such that

$$\sqrt{\frac{1^2 + 2^2 + \dots + (n-1)^2 + n^2}{n}}$$

is a positive integer. (Hint: Reduce this problem to Pell's equation using the formula $1^2 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6$.)

If $m = \sqrt{\frac{1^2 + 2^2 + \dots + n^2}{n}}$, then the hint yields that $m^2 = (n+1)(2n+1)/6$. Multiplying by 48 yields

$$48m^2 = 16n^2 + 24n + 8 = (4n + 3)^2 - 1.$$

This can be written as $x^2 - 48y^2 = 1$ for $x = 4n + 3$ and $y = m$. This is an instance of Pell's equation.

Solution 1. The continued fraction for $\sqrt{48}$ is $[6; \overline{1, 12}]$. (You can try to determine this for yourself, but it is also possible to use a continued fractions calculator, like the one at <https://www.wolframalpha.com/input/?i=continued+fraction>) We seek a solution to $x^2 - 48y^2 = 1$ where $x = 4n + 3$ by examining the convergents $\frac{p_k}{q_k}$ of $[6; \overline{1, 12}]$ so that we can choose $(x, y) = (p_k, q_k)$. We want $x = 4n + 3$, so we seek a convergent where $p_k \equiv 3 \pmod{4}$.

The first few convergents of $[6; \overline{1, 12}]$ are

$$\frac{6}{1}, \frac{7}{1}, \frac{90}{13}, \frac{97}{14}, \frac{1254}{181}, \frac{1351}{195}.$$

We can stop looking now, because the numerator of $\frac{p_5}{q_5} = \frac{1351}{195}$ satisfies

$$p_5 = 1351 \equiv 3 \pmod{4}.$$

Hence, if we take $x = 1351 = 4n + 3$ and $y = 195 = m$, we derive that $n = 337$. This yields that the quadratic mean of $1, 2, 3, \dots, 337$ is the integer 195.

Solution 2.¹ The equation $(4n + 3)^2 - 48m^2 = 1$ can be reduced further by noticing that $48m^2 = 3 \cdot 4^2 \cdot m^2$, hence it can be written $x^2 - 3y^2 = 1$ where $x = 4n + 3$ and $y = 4m$. Thus, we look for solutions (x, y) to $x^2 - 3y^2 = 1$ where $x \equiv 3 \pmod{4}$ and $y \equiv 0 \pmod{4}$.

We know the continued fraction expansion for $\sqrt{3}$ to be

$$\sqrt{3} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}}$$

so $\sqrt{3} = [1; \overline{1, 2}]$. Its convergents are $1, 2/1, 5/3, 7/4, \dots$. The first convergent that solves Pell's equation and also satisfies $x \equiv 3 \pmod{4}$ and $y \equiv 0 \pmod{4}$ is $x = 7, y = 4$. In this case, $n = 1, m = 1$. But we seek a solution with $n > 1$, so we keep examining convergents:

$$19/11, 26/15, 71/41, 97/56, 265/153, 362/209, 989/571, 1351/780.$$

We can stop here, since the denominator $780 = y = 4m$ is divisible by 4 and the numerator $1351 = x = 4n + 3$ is congruent to 3 modulo 4. Solving for m and n yields $m = 780/4 = 195$ and $n = (1351 - 3)/4 = 337$. This means that

$$\sqrt{\frac{1^2 + 2^2 + \dots + 337^2}{337}} = 195.$$

Thus, $n = 337$ is the first $n > 1$ such that the quadratic mean of the first n integers is again an integer. **The next example is**

$$\sqrt{\frac{1^2 + 2^2 + \dots + 65521^2}{65521}} = 37829.$$

¹The advantage of Solution 2 over Solution 1 is that the d -value in $x^2 - dy^2 = 1$ is smaller in Solution 2 ($d = 3$) than in Solution 1 ($d = 48$). Both solution methods require knowing the continued fraction expansion of \sqrt{d} . We already worked out the continued fraction expansion of $\sqrt{3}$ in class, while we have not yet worked out the continued fraction expansion of $\sqrt{48}$. The disadvantage of Solution 2 is that we must calculate more convergents until we reach a solution. (We calculate until the 5th convergent in Solution 1 and to the 11th convergent in Solution 2.)