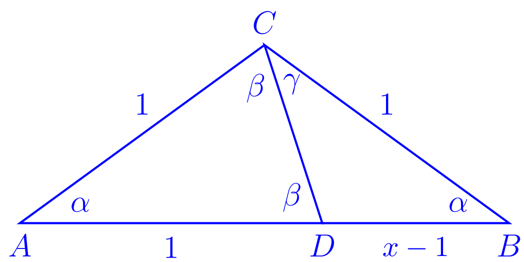
HW 5: solution sketches

(1) Exercise 2.3.3.



We will use only the top triangle of Figure 2.6 of the text. Label the vertices A, B, C, as above. Now add another point D on $\overline{AB}$ that is distance 1 from A and distance $x-1$ from B. Since $\angle ACB$ is an angle of a regular pentagon, it measures $108°$. Since $\triangle ABC$ is isosceles with vertex C, $\angle CAB = \angle CBA = \alpha$, and so $\alpha+\alpha+108° = 180°$, which leads to $\alpha = 36°$. Since $\triangle ACD$ is isosceles with vertex A, a similar calculation shows that $\beta = 72°$. Now we calculate that $\gamma = (\angle ACB)° - \beta = 108° - 72° = 36°$. We conclude that $\gamma = \alpha$, so $\triangle BCD$ is similar to $\triangle ABC$. Since they are similar, ratios of corresponding side lengths ($\overline{AB}$ to $\overline{AC}$ versus $\overline{BC}$ to $\overline{BD}$) are equal. We conclude that $x/1 = 1/(x-1)$.

(2) Exercise 2.3.4.

From Exercise 2.3.3, the length, x, of the diagonal of the regular pentagon in Figure 2.6 satisfies $x/1 = 1/(x-1)$, or equivalently $x^2 - x - 1 = 0$. The number x must be the positive root of $x^2 - x - 1 = 0$, so, by the quadratic formula, $x = \frac{1+\sqrt{5}}{2}$. This is a constructible number, so we can construct x, hence we can construct all three sides $1, 1, x$ of the upper triangle in Figure 2.6. Given three sides of a triangle, we can construct the triangle itself. Hence we can construct all the angles of the triangle. This yields a construction of the top angle in Figure 2.6, $\frac{3\pi}{5}$, and once that angle has been constructed we can copy it over and over in the right places to produce a regular pentagon.

(3) Verify the correctness of the construction of the regular pentagon indicated in the gif above.

Let me name some of the points in the construction.

(a) Let C be the center of the circle. (Assume the coordinates of C are at $(0,0)$ and that the circle has radius 1.)
(b) Let A and B be the points at $(0,-1)$ and $(0,1)$, which are at the endpoints of the vertical diameter.
(c) Let D be the point at $(-1,0)$ that is constructed next.
(d) Let E be the point at $(-\frac{1}{2},0)$ that is constructed by bisecting the segment $\overline{CD}$.

(e) Let F be the point of intersection of the segment $\overline{CB}$ and the bisector of angle $\angle BEC$.

(f) Let G be the point of intersection of the circle and the line through F that is perpendicular to $\overline{CB}$.

The main calculation of this problem is to show that the length $\underline{x}$ of $\overline{CF}$ is $\cos\left(\frac{2\pi}{5}\right) = \cos(72°)$. This will show that the angle $\angle BCG$ is $\frac{2\pi}{5}$, thereby showing that the arc from B to G is one fifth of the way around the circle. Having constructed this, we can mark five equally spaced points on the circle and construct the regular pentagon.

I am going to need the half-angle formula:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

when $\theta = \angle CEB$. Since the length of $\overline{CB}$ is 1 and the length of $\overline{CE}$ is $\frac{1}{2}$, we get from the Pythagorean Theorem that the length of $\overline{EB}$ is $\frac{\sqrt{5}}{2}$. Hence $\cos(\angle CEB)$ is $(1/2)/(\sqrt{5}/2) = 1/\sqrt{5}$. From this we get that

$$\begin{aligned}(2x =)\quad x/(1/2) &= \tan(\angle CEF)\\ &= \tan\left(\tfrac{1}{2}\angle CEB\right)\\ &= \sqrt{\tfrac{1-\cos(\angle CEB)}{1+\cos(\angle CEB)}}\\ &= \sqrt{\tfrac{1-1/\sqrt{5}}{1+1/\sqrt{5}}}\\ &= \sqrt{\tfrac{\sqrt{5}-1}{\sqrt{5}+1}}\\ &= \sqrt{\tfrac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)(\sqrt{5}-1)}}\\ &= \tfrac{(\sqrt{5}-1)}{2}\end{aligned}$$

so $x = \frac{\sqrt{5}-1}{4}$. Our goal was to show that $x = \cos\left(\frac{2\pi}{5}\right)$, so it remains to show that $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$.

If $\omega = \cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)$, then ω is a root of $0 = x^5-1 = (x-1)(x^4+x^3+x^2+x+1)$. Since ω is not a root of $0 = x-1$ it must be a root of $0 = x^4+x^3+x^2+x+1$. This implies that $\omega^4+\omega^3+\omega^2+\omega+1 = 0$, hence (dividing by ω^2) we derive that

$$\omega^2+\omega+1+\omega^{-1}+\omega^{-2} = 0.$$

Let's rewrite this fact in terms of $\Omega = \omega + \omega^{-1} = 2\cos\left(\frac{2\pi}{5}\right)$:

$$\begin{aligned} 0 &= \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} \\ &= (\omega^2 + 2 + \omega^{-2}) + \omega + \omega^{-1} - 1 \\ &= \Omega^2 + \Omega - 1. \end{aligned}$$

This shows that $\Omega (= 2\cos\left(\frac{2\pi}{5}\right))$ is a (positive) root of $y^2 + y - 1 = 0$. By the quadratic formula, the roots of this equation are $\frac{-1\pm\sqrt{5}}{2}$. The only positive root is

$$\frac{-1+\sqrt{5}}{2} = \left(= 2\cos\left(\frac{2\pi}{5}\right)\right),$$

so $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$, as desired.