HW 3: solution sketches

(1) Use the Pythagorean theorem to show that $AB = BC = CA$ in Figure 2.2.

Using the ideas in the next solution, the coordinates of the points A, B and C are:

$$A = \frac{1}{2}(1, 0, \tau), B = \frac{1}{2}(\tau, -1, 0), C = \frac{1}{2}(\tau, 1, 0).$$

The leading $\frac{1}{2}$ that occurs here is due to the fact that Golden Rectangles of Exercise 2.2.2 have dimensions that are half of those of Exercise 2.2.3.

Since $C - B = (0, 1, 0)$, we have $|BC| = \sqrt{0^2 + 1^2 + 0^2} = 1$. Since $A - B = \frac{1}{2}(1 - \tau, 1, \tau)$, and since $\tau^2 = \tau + 1$, we have

$$\begin{aligned} |AB| &= \tfrac{1}{2}\sqrt{(1-\tau)^2 + 1^2 + \tau^2} \\ &= \tfrac{1}{2}\sqrt{2 - 2\tau + 2\tau^2} \\ &= \tfrac{1}{2}\sqrt{2 - 2\tau + 2(1+\tau)} \\ &= \tfrac{1}{2}\sqrt{4} = \tfrac{1}{2} \cdot 2 = 1. \end{aligned}$$

A symmetric argument shows that $|AC| = 1$. Altogether, we have $|AB| = |AC| = |BC| = 1$.

(2) Show that the coordinates of the vertices of the icosahedron are $(\pm 1, 0, \pm\tau)$, $(\pm\tau, \pm 1, 0)$, and $(0, \pm\tau, \pm 1)$, for all possible combinations of $+$ and $-$ signs.

Imagine that the origin $(0, 0, 0)$ of a 3-dimensional coordinate system is located where the three rectangles in Figure 2.2 meet. In Figure 2.2, at the point of the origin, 3 axes appear to emanate; these will be the x-, y-, and z-axes. Consider the horizontal rectangle to be the xy-plane with point B having positive x-coordinate and negative y-coordinate, while point C has positive x-coordinate and positive y-coordinate. Consider the vertical direction to be z-direction, with "up" being the positive z-direction.

The Golden Rectangles of Figure 2.2 all have dimensions $2 \times 2\tau$. These rectangles lie in (or represent) the xy-, yz-, and xz-axes, with the longer dimension of the rectangles lying along the x-, y-, and z-axes respectively. The Golden Rectangle representing the xy-plane is centered at the origin and its longer dimension is along the x-axis, so its vertices must have coordinates at $(\pm\tau, \pm 1, 0)$. Similarly, the Golden Rectangle representing the yz-plane must have vertices at $(0, \pm\tau, \pm 1)$. Finally, the Golden Rectangle representing the xz-plane must have vertices at $(\pm 1, 0, \pm\tau)$.

(3) Find the total defect of each of the Platonic solids.

We can calculate total defect using the data from the Jan 30 handout by multiplying the value in the Vertices column by the value in the Defect/vertex column:

Platonic Solid	#Vertices/Face	# Vertices	# Edges	# Faces	Defect/vertex	Total Defect
Tetrahedron	3	4	6	4	180°	720°
Cube	4	8	12	6	90°	720°
Octahedron	3	6	12	8	120°	720°
Dodecahedron	5	20	30	12	36°	720°
Icosahedron	3	12	30	20	60°	720°