HW 2: solution sketches

(1) Give a geometric proof that $\sqrt{3}$ is irrational. (Hint: It might be easier to show that $1+\sqrt{3}$ is irrational, then deduce that $\sqrt{3}$ is also irrational.)

Start with a rectangle of height 1 and base $1+\sqrt{3}$. Delete two maximal subsquares (which are 1×1 squares) to obtain a remainder rectangle of base $(1+\sqrt{3})-2$ (which is less than 1) and height 1. Now delete one maximal subsquare of base $(1+\sqrt{3})-2=\sqrt{3}-1$ to obtain a remainder rectangle of base $\sqrt{3}-1$ and height $1-(\sqrt{3}-1)=2-\sqrt{3}$. The current rectangle is similar to the starting rectangle, since

$$(1+\sqrt{3})/1=(\sqrt{3}-1)/(2-\sqrt{3}).$$

This implies that if you keep deleting maximal subsquares, the process will not terminate, so the starting values 1 and $1+\sqrt{3}$ are not commensurable. Equivalently, $1+\sqrt{3}$ is not rational.

Temporarily assume that $\sqrt{3}$ IS rational. Since 1 is rational, and the sum of rationals is rational, it follows from our temporary assumption that $1+\sqrt{3}$ is rational. Since we proved the irrationality of $1+\sqrt{3}$ in the previous paragraph, our assumption that $\sqrt{3}$ is rational must be false. We conlclude that $\sqrt{3}$ is irrational.

(2) Use the Euclidean algorithm to find an integral solution to $270x+168y=6$.

First use the Euclidean algorithm:

$$\begin{aligned}
270 &= 168 \cdot 1 + 102 \\
168 &= 102 \cdot 1 + 66 \\
102 &= 66 \cdot 1 + 36 \\
66 &= 36 \cdot 1 + 30 \\
36 &= 30 \cdot 1 + 6 \\
30 &= 6 \cdot 5 + 0
\end{aligned}$$

Rewriting this data yields $6=(36-30\cdot 1)=(36-(66-36\cdot 1)\cdot 1)=(36\cdot 2-66\cdot 1)=\cdots=\underline{\mathbf{5}}\cdot 270-\underline{\mathbf{8}}\cdot 168$. Our integral solution is $(x,y)=(5,-8)$.

(3) What is the height of a regular tetrahedron of side length 1?

There are various ways to do this, but the simplest might be to find a nice regular tetrahedron in space (not necessarily of side length 1), and then compute the ratio of the height to the side length.

There is a nice regular tetrahedron in 4-space, namely the one with vertices at $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$. If the first three define the bottom face, then its center is at $(e_1+e_2+e_3)/3 = (1/3,1/3,1/3,0)$. The distance from the top, e_4, to this point is $\|e_4 - (e_1+e_2+e_3)/3\|$, or

$$\sqrt{(1/3)^2+(1/3)^2+(1/3)^2+1} = 2\sqrt{3}/3.$$

The side length of this tetrahedron is the length from e_1 to e_2, which is $\|e_1 - e_2\| = \sqrt{2}$. The desired ratio is now $(2\sqrt{3}/3)/\sqrt{2} = \sqrt{6}/3 = \sqrt{2/3}$.