HW 1: solution sketches

(1) Exercise 1.4.2.

In Figure 1.9, denote the height of the triangle by h. The outer right triangle with sides $a, b, c_1 + c_2$ shares an acute angle with each of the two smaller triangles (one smaller triangle has hypotenuse a and legs h, c_1 while the other smaller triangle has hypotenuse b and legs h, c_2). This is enough to show that all three triangles are similar to the outer triangle, because it forces corresponding angles to be equal. Ratios of corresponding sides must be equal, so we get $a/c_1 = (c_1 + c_2)/a$, which may be rewritten $a^2 = c_1^2 + c_1 c_2$. A similar calculation with the other smaller triangle yields $b^2 = c_2^2 + c_1 c_2$. Adding these yields $a^2 + b^2 = c_1^2 + 2c_1c_2 + c_2^2 = (c_1 + c_2)^2 = c^2$.

(2) True or False? Every integer $n > 2$ occurs in some Pythagorean Triple. (Justify your answer.)

TRUE.

Every primitive Pythagorean Triple has the form $(a, b, c) = (p^2 - q^2, 2pq, p^2 + q^2)$ where $p > q > 0$, $\gcd(p, q) = 1$ and p and q have opposite parity (meaning that one is even and the other is odd). Moreover, any triple of the form $(p^2 - q^2, 2pq, p^2 + q^2)$ with $p > q > 0$ is a Pythagorean Triple, even if $\gcd(p, q) \neq 1$ or p and q have the same parity. (This statement means only that each term in the triple $(p^2 - q^2, 2pq, p^2 + q^2)$ is positive and $(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$.)

To show that any $n > 2$ appears in a triple of the form $(p^2 - q^2, 2pq, p^2 + q^2)$ it suffices to note that if $n = 2k$ is even, then we can take $p = k$ and $q = 1$. Then $n = 2k = 2pq$, appears as the middle term of $(p^2 - q^2, 2pq, p^2 + q^2)$ and $p > q > 0$.

Now if $n > 2$ is odd, we want to arrange that $n = p^2 - q^2$ for some $p > q$. That is, we want $n = p^2 - q^2 = (p + q)(p - q)$. Using the fact that n is odd, and equating factorizations $n = n \cdot 1 = (p + q)(p - q)$, we can take $n = p + q$ and $1 = p - q$, solve for p and q, and obtain $p = (n + 1)/2 \in \mathbb{Z}$ and $q = (n - 1)/2 \in \mathbb{Z}$. Thus, for $p = (n + 1)/2$ and $q = (n - 1)/2$ we have that $p > q > 0$ and the first term in $(p^2 - q^2, 2pq, p^2 + q^2)$ is n.

(3) Explain why there are only finitely many distinct Pythagorean Triples (a, b, c) with $a = 100$.

By the parametrization of the Pythagorean Triples, we know that each such PT (a, b, c) has the form $a = (p^2 - q^2)r$, $b = 2pqr$, and $c = (p^2 + q^2)r$ for some positive integers p, q, r with $p > q$. When $a = 100$ we have $100 = (p^2 - q^2)r = (p - q)(p + q)r$.

Assume instead that there are infinitely many distinct PTs with $a = 100$, say

$$(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), \ldots \tag{0.1}$$

are distinct, but $100 = a_1 = a_2 = \cdots$. The list (0.1) leads to infinitely many factorizations of the form $100 = a_i = (p_i - q_i)(p_i + q_i)r_i$. But 100 can be factored into three factors in only finitely many different ways, so we must have $i < j$ where (a_i, b_i, c_i) and (a_j, b_j, c_j) are different PTs, but the sequences of factors $((p_i - q_i), (p_i + q_i), r_i)$ and $((p_j - q_j), (p_j + q_j), r_j)$ are the same. This implies that $p_i = \frac{(p_i - q_i) + (p_i + q_i)}{2} = \frac{(p_j - q_j) + (p_j + q_j)}{2} = p_j$, $q_i = \frac{(p_i + q_i) - (p_i - q_i)}{2} = \frac{(p_j + q_j) - (p_j - q_j)}{2} = q_j$, and $r_i = r_j$. But now, since $(p_i, q_i, r_i) = (p_j, q_j, r_j)$ we get that $(a_i, b_i, c_i) = ((p_i^2 - q_i^2)r_i, 2p_i q_i r_i, (p_i^2 + q_i^2)r_i) = ((p_j^2 - q_j^2)r_j, 2p_j q_j r_j, (p_j^2 + q_j^2)r_j) = (a_j, b_j, c_j)$, contradicting our assumption that (a_i, b_i, c_i) and (a_j, b_j, c_j) are distinct PTs.