Cosets of Subgroups

Cosets of subgroups

Definition.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.)

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form
$Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2 f\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\} = (rf)S$.
   3. $(r^2)S = \{r^2, r^2f\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \ \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2f\} = (r^2f)S$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2 f\} = (r^2 f)S$.
    4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2 f\}\}$

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2f\} = (r^2f)S$.
    4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2f\}\}$
2. (Right cosets of S in G)

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\} = (rf)S$.
   3. $(r^2)S = \{r^2, r^2f\} = (r^2f)S$.
   4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2f\}\}$
2. (Right cosets of S in G)
   1. $S1 = Sf = \{1, f\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2 f\} = (r^2 f)S$.
    4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2 f\}\}$
2. (Right cosets of S in G)
    1. $S1 = Sf = \{1, f\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2f\} = (r^2f)S$.
    4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2f\}\}$
2. (Right cosets of S in G)
    1. $S1 = Sf = \{1, f\}$.
    2. $Sr = S(r^2f) = \{r, r^2f\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2f\} = G$.

1. (Left cosets of S in G)
    1. $1S = \{1, f\} = fS$.
    2. $rS = \{r, rf\} = (rf)S$.
    3. $(r^2)S = \{r^2, r^2f\} = (r^2f)S$.
    4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2f\}\}$
2. (Right cosets of S in G)
    1. $S1 = Sf = \{1, f\}$.
    2. $Sr = S(r^2f) = \{r, r^2f\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\} = (rf)S$.
   3. $(r^2)S = \{r^2, r^2 f\} = (r^2 f)S$.
   4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2 f\}\}$
2. (Right cosets of S in G)
   1. $S1 = Sf = \{1, f\}$.
   2. $Sr = S(r^2 f) = \{r, r^2 f\}$.
   3. $S(r^2) = S(rf) = \{r^2, rf\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\} = (rf)S$.
   3. $(r^2)S = \{r^2, r^2 f\} = (r^2 f)S$.
   4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2 f\}\}$
2. (Right cosets of S in G)
   1. $S1 = Sf = \{1, f\}$.
   2. $Sr = S(r^2 f) = \{r, r^2 f\}$.
   3. $S(r^2) = S(rf) = \{r^2, rf\}$.

Cosets of subgroups

Definition. Let G be a group and let S be a subgroup. A **left coset** of S in G is a set of the form $gS = \{gs \in G \mid s \in S\}$, $g \in G$. (Here $g \in G$ is fixed and s varies over S.) A **right coset** of S in G is a set of the form $Sg = \{sg \in G \mid s \in S\}$, $g \in G$.

Examples. Assume that $S = \{1, f\} \leq D_3 = \{1, r, r^2, f, rf, r^2 f\} = G$.

1. (Left cosets of S in G)
   1. $1S = \{1, f\} = fS$.
   2. $rS = \{r, rf\} = (rf)S$.
   3. $(r^2)S = \{r^2, r^2 f\} = (r^2 f)S$.
   4. (Set of all left cosets) $G/S = \{\{1, f\}, \{r, rf\}, \{r^2, r^2 f\}\}$
2. (Right cosets of S in G)
   1. $S1 = Sf = \{1, f\}$.
   2. $Sr = S(r^2 f) = \{r, r^2 f\}$.
   3. $S(r^2) = S(rf) = \{r^2, rf\}$.
   4. (Set of all right cosets) $S\backslash G = \{\{1, f\}, \{r, r^2 f\}, \{r^2, rf\}\}$

Cosets of subgroups

Theorem.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i),

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$,

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$,

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii),

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\}$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g\in G}\{g\} \subseteq \underline{\bigcup gS}$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$,

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii),

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$,

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then b

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g\in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$.

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h)

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h) shows that $hS \subseteq gS$.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h) shows that $hS \subseteq gS$. Hence $gS = hS$.

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G} \{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h) shows that $hS \subseteq gS$. Hence $gS = hS$. $\square$

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h) shows that $hS \subseteq gS$. Hence $gS = hS$. □

Similarly, the right cosets of S partition G.

Cosets of subgroups

Theorem. If G is a group and $S \leq G$, then the set of left cosets of S is a partition of G.

Proof. We must show that $G/S = \{gS \mid g \in G\}$ is a (i) set of nonempty sets (ii) whose union is G and (iii) where any two are disjoint or equal.

For (i), if $gS \in G/S$, then $g = g \cdot 1 \in gS$, so $gS \neq \emptyset$.

For (ii), $G = \bigcup_{g \in G}\{g\} \subseteq \underline{\bigcup gS} \subseteq G$, hence $\bigcup gS = G$.

For (iii), assume that $gS \cap hS \neq \emptyset$. Choose $a \in gS \cap hS$. There exist $u, v \in S$ such that $gu = a = hv$. Let's explain why $gS \subseteq hS$. Choose any $b \in gS$, say $b = gw$ for some $w \in S$. Then $b = gw = (hvu^{-1})w = h(vu^{-1}w) \in hS$. This shows that $gS \subseteq hS$. A symmetric argument (interchanging the roles of g and h) shows that $hS \subseteq gS$. Hence $gS = hS$. $\square$

Similarly, the right cosets of S partition G.

The partition of G into left cosets of S may be different from the partition of G into right cosets of S.

The size and number of cosets of a subgroup

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But!

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S \backslash G|$.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S \backslash G|$.

Proof. For the first item,

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a : S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S|$

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a : S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S|$ (= $|Sb|$).

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a : S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S|$ (= $|Sb|$).

For the second item,

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a : S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S| \quad (= |Sb|)$.

For the second item, verify that **inversion**,

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a\colon S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S| \quad (= |Sb|)$.

For the second item, verify that **inversion**,

$$I\colon G/S \to S\backslash G\colon gS \mapsto (gS)^{-1} = S^{-1}g^{-1} = Sg^{-1},$$

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S \backslash G|$.

Proof. For the first item, verify that $\lambda_a \colon S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S| \quad (= |Sb|)$.

For the second item, verify that **inversion**,

$$I \colon G/S \to S \backslash G \colon gS \mapsto (gS)^{-1} = S^{-1}g^{-1} = Sg^{-1},$$

is a bijection

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a \colon S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S|$ (= $|Sb|$).

For the second item, verify that **inversion**,

$$I \colon G/S \to S\backslash G \colon gS \mapsto (gS)^{-1} = S^{-1}g^{-1} = Sg^{-1},$$

is a bijection whose inverse is also inversion.

The size and number of cosets of a subgroup

(Repeat) The partition of G into left cosets of a subgroup S may be different from the partition of G into right cosets of S. But! the size and number of left cosets of S in G are the same as the size and number of right cosets of S in G.

Theorem. Let S be a subgroup of a group G.

1. For any $a, b \in G$, $|aS| = |Sb|$.
2. $|G/S| = |S\backslash G|$.

Proof. For the first item, verify that $\lambda_a : S \to aS$ is a bijection with inverse $\lambda_{a^{-1}}$. Hence $|aS| = |S|$ (= $|Sb|$).

For the second item, verify that **inversion**,

$$I: G/S \to S\backslash G: gS \mapsto (gS)^{-1} = S^{-1}g^{-1} = Sg^{-1},$$

is a bijection whose inverse is also inversion. □

The index of a subgroup

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition**

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$.

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same.

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same. ($|G/S| = |S\backslash G|$.)

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same. ($|G/S| = |S\backslash G|$.) Let's introduce terminology and notation for this number:

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same. ($|G/S| = |S\backslash G|$.) Let's introduce terminology and notation for this number:

Definition.

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same. ($|G/S| = |S\backslash G|$.) Let's introduce terminology and notation for this number:

Definition. The **index** of S in G is the number of (left or right) cosets of S in G. Write $[G\colon S]$ or $|G\colon S|$ to denote the index of S in G.

The index of a subgroup

The first part of the previous theorem implies that the partition of G into left cosets of S is a **uniform partition** (= all cells have the same size), and that the common size of the cells is $|S|$. The same is true for the partition of G into right cosets of S.

The second part of the theorem implies that the number of cells in each of the partitions G/S and $S\backslash G$ is the same. ($|G/S| = |S\backslash G|$.) Let's introduce terminology and notation for this number:

Definition. The **index** of S in G is the number of (left or right) cosets of S in G. Write $[G\colon S]$ or $|G\colon S|$ to denote the index of S in G.

When we say "the number of cosets" we mean "the size of a **transversal**" or "the size of a **system of distinct representatives**."

Index formula

Index formula

“Cow Theorem”.

Index formula

“Cow Theorem”. If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G \colon S] \cdot |S|$.

When G and S are finite, this implies that $[G \colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G \colon S]$

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2f, \ldots, r^{n-1}f\}$. $[G\colon S] = |S\backslash G|$

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \dots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \dots, r^{n-1}f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \dots, r^{n-1}f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\}$

Index formula

“Cow Theorem”. If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S \backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1} f, r^{-2} f, \ldots, r^{1-n} f\}$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S \backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1} f, r^{-2} f, \ldots, r^{1-n} f\}$. $[G\colon S]$

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1} f, r^{-2} f, \ldots, r^{1-n} f\}$. $[G\colon S] = |G/S|$

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S \backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1} f, r^{-2} f, \ldots, r^{1-n} f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1}f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1}f, r^{-2}f, \ldots, r^{1-n}f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.
3. Of course, $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ and $2\mathbb{Z}$ are infinite, we do not assign meaning to $|\mathbb{Z}|/|2\mathbb{Z}|$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1} f, r^{-2} f, \ldots, r^{1-n} f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.
3. Of course, $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ and $2\mathbb{Z}$ are infinite, we do not assign meaning to $|\mathbb{Z}|/|2\mathbb{Z}|$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S \backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1}f, r^{-2}f, \ldots, r^{1-n}f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.
3. Of course, $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ and $2\mathbb{Z}$ are infinite, we do not assign meaning to $|\mathbb{Z}|/|2\mathbb{Z}|$. Nevertheless, we do assign meaning to $[\mathbb{Z}\colon 2\mathbb{Z}]$.

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2f, \ldots, r^{n-1}f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1}f, r^{-2}f, \ldots, r^{1-n}f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.
3. Of course, $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ and $2\mathbb{Z}$ are infinite, we do not assign meaning to $|\mathbb{Z}|/|2\mathbb{Z}|$. Nevertheless, we do assign meaning to $[\mathbb{Z}\colon 2\mathbb{Z}]$. $[\mathbb{Z}\colon 2\mathbb{Z}] = 2$,

Index formula

"Cow Theorem". If G is a group and $S \leq G$, then $|G| = [G\colon S] \cdot |S|$.

When G and S are finite, this implies that $[G\colon S] = |G|/|S|$.

Examples.

1. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $S\backslash G = \{S, Sf\}$ where $Sf = \{f, rf, r^2 f, \ldots, r^{n-1} f\}$. $[G\colon S] = |S\backslash G| = |\{S, Sf\}| = 2$.
2. $G = D_n$, $S = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. $G/S = \{S, fS\}$ where $Sf = \{f, fr, fr^2, \ldots, fr^{n-1}\} = \{f, r^{-1}f, r^{-2}f, \ldots, r^{1-n}f\}$. $[G\colon S] = |G/S| = |\{S, fS\}| = 2 = |G|/|S|$.
3. Of course, $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ and $2\mathbb{Z}$ are infinite, we do not assign meaning to $|\mathbb{Z}|/|2\mathbb{Z}|$. Nevertheless, we do assign meaning to $[\mathbb{Z}\colon 2\mathbb{Z}]$. $[\mathbb{Z}\colon 2\mathbb{Z}] = 2$, since $\mathbb{Z}/2\mathbb{Z} = \{2\mathbb{Z}, 1 + 2\mathbb{Z}\}$.

Normal subgroups

Normal subgroups

Definition.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \triangleleft G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \triangleleft G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \text{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.
2. $\langle r\rangle \lhd D_n$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.
2. $\langle r \rangle \lhd D_n$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \text{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.
2. $\langle r \rangle \lhd D_n$.
3. $\langle f \rangle \ntriangleleft D_n$ if $n > 2$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \text{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.
2. $\langle r \rangle \lhd D_n$.
3. $\langle f \rangle \ntriangleleft D_n$ if $n > 2$.

Normal subgroups

Definition. If G is a group and $N \leq G$, then N is a **normal** subgroup of G if $G/N = N\backslash G$. Write $N \lhd G$ to denote this.

The following are equivalent conditions about $N \leq G$.

1. $N \lhd G$. (That is, $G/N = N\backslash G$.)
2. $gN = Ng$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.
4. $g^{-1}Ng = N$ for all $g \in G$.
5. $(aN)(bN) = (ab)N$ for all $a, b \in G$.
6. $N = \mathrm{Ker}(h)$ for some homomorphism h with domain G.

Examples/Nonexamples.

1. Every subgroup of an abelian group is normal.
2. $\langle r \rangle \lhd D_n$.
3. $\langle f \rangle \ntriangleleft D_n$ if $n > 2$.