Continued Fractions and Pell's Equation

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4. If we truncate $[a_0; a_1, a_2, \ldots]$ at the n-th term $[a_0; a_1, a_2, \ldots, a_n]$, the resulting fraction $\frac{h_n}{k_n}$ is called the n-th convergent of $[a_0; a_1, a_2, \ldots]$.

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Theorem. If $[a_0; a_1, \ldots]$ is irrational and there is a real number $r > 1$ such that $a_{n+1} \geq r \cdot a_n^n$ for almost all n, then $[a_0; a_1, \ldots]$ is transcendental.

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It turns out that if d is an nonsquare positive integer, then the continued fraction expansion of $\sqrt{d}$ has a "palindromic block"

$$\sqrt{d} = [a_0; a_1, a_2, \ldots, a_2, a_1, 2a_0, a_1, \ldots].$$

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2. (Optimal rational approximation) If $\alpha = [a_0; a_1, a_2, \ldots]$ is irrational, then $\frac{h_n}{k_n}$ is an optimal rational approximation to α in the sense that if $\frac{h}{k}$ is another rational approximation with $1 \leq k \leq k_n$, then

$$\left|\alpha - \frac{h_n}{k_n}\right| \leq \left|\alpha - \frac{h}{k}\right|.$$

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Open Problem

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Problem. (Waldschmidt, 2004)

Open Problem

Problem. (Waldschmidt, 2004) Does there exist a real algebraic number $\alpha = [a_0; a_1, a_2, \ldots]$ whose degree is ≥ 3 where the sequence $(a_0, a_1, a_2, \ldots)$ is bounded?