The Field of Real Constructible Numbers.

The field K of real constructible numbers is the smallest Euclidean subfield of $\mathbb{C}$, which means that it is the smallest subfield of $\mathbb{C}$ closed under square roots of positive real numbers. To show that a number $z \in \mathbb{C}$ belongs to K, write down a construction sequence. To show that $z \notin K$, one can use the following theorem.

Theorem 1. *A number $z \in K$ has the following properties.*

(1) *z is real.*
(2) *z is* algebraic. *(This means that z is a root of a nonzero rational polynomial.)*
(3) *The minimal polynomial of z over $\mathbb{Q}$, $\min_{z,\mathbb{Q}}(x)$ has degree that is a power of 2.*
(4) *The size of the Galois group of $\min_{z,\mathbb{Q}}(x)$ is a power of 2.*
(5) *Any subcollection $H \subseteq \mathrm{Gal}(\min_{z,\mathbb{Q}}(x))$ that is closed under multiplication has size that is a power of 2.*

Arithmetic Consequences.

(1) $\sqrt{\pi} \notin K$. ($\sqrt{\pi}$ is not algebraic.)
(2) $\sqrt[3]{2} \notin K$. ($\min_{\sqrt[3]{2},\mathbb{Q}}(x) = x^3 - 2$ has degree 3, which is not a power of 2.)
(3) $\cos(2\pi/n) \in K$ iff $\phi(n)$ is a power of 2.

Geometric Consequences.

(1) It is impossible to square the circle with straightedge and compass.
(2) It is impossible to duplicate the cube with straightedge and compass.
(3) An angle of the form $2\pi/n$ is constructible with straightedge and compass iff $\phi(n)$ is a power of 2. In particular, $2\pi/3$ is constructible, while $2\pi/9$ is not.