

Linear change of coordinates in \mathbb{RP}^2 .

Recall that

(1) We take the points of \mathbb{RP}^2 to be the lines through the origin in \mathbb{R}^3 excluding $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(2) We write points of \mathbb{RP}^2 in coordinates in the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, but points are only determined up to nonzero scaling:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \sim \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \left(= \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \end{bmatrix} \right), \quad \lambda \neq 0.$$

(3) We take the lines of \mathbb{RP}^2 to be the planes through the origin in \mathbb{R}^3 — $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(4) We write the line of \mathbb{RP}^2 whose equation is $Ax + By + Cz = 0$ as $[A \ B \ C]$. Lines are only determined up to nonzero scaling.

(5) A point $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is incident to a line $[A \ B \ C]$ iff $0 = [A \ B \ C] \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = Aa + Bb + Cc$.

(6) If $z \neq 0$, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a point in the “finite part of the plane” ($\cong \mathbb{R}^2$). After scaling,

this point may be written $\begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$, and its Euclidean coordinates are $\begin{bmatrix} x/z \\ y/z \end{bmatrix}$.

(7) If $F(x, y)$ is a polynomial of degree k and $F(x, y) = 0$ defines a curve C in \mathbb{R}^2 , then the **homogeneous polynomial equation** $z^k F(x/z, y/z) = 0$ defines the **projective completion** of C in \mathbb{RP}^2 .

(8) The group $\text{GL}_3(\mathbb{R})$ of invertible 3×3 real matrices acts on \mathbb{RP}^2 , and each of the matrices in this group defines a linear change of variables for \mathbb{RP}^2 .

(9) If $M = \begin{bmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{bmatrix} \in \text{GL}_3(\mathbb{R})$, then the change of variables determined by M will

map $X \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $Y \mapsto \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$, $O \mapsto \begin{bmatrix} a'' \\ b'' \\ c'' \end{bmatrix}$, where X is the point at infinity on the x -axis, Y is the point at infinity on the y -axis, and O is the origin.

- (10) If $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $F(\mathbf{x}) = 0$ is a homogeneous polynomial equation that defines a curve C , and $M \in \text{GL}_3(\mathbb{R})$ is a linear change of basis matrix, then $F(M^{-1}\mathbf{x}) = 0$ expresses C in the new basis.

Exercises.

- (1) Write down the homogeneous form of each of the following polynomial equations:

(a) $y = x^2$

(b) $x^2 + y^2 = 1$

(c) $2xy = 1$

- (2) Sketch the three curves defined by these equations.

- (3) For each of the above curves, find a linear change of variables that maps the curve entirely into the finite part of the plane. What does the equation for the curve reduce to when $z = 1$?