Challenge Problem: Find all Pythagorean triples that form an arithmetic progression.

Solution:

Suppose that positive integers a, b, c satisfy (i) $a^2 + b^2 = c^2$ and (ii) a, b and c are in arithmetic progression. Since c is necessarily the largest, we may assume that the progression is $a < b < c$.

Stage 1. Reduce to a 'primitive' Pythagorean triple.

Notice that if a and b have a common factor, say d, then $a = a_1 \cdot d, b = b_1 \cdot d$ for some a_1 and b_1. Hence $c^2 = a^2 + b^2 = (a_1^2 + b_1^2)d^2$. This forces $d \mid c$, so we may write $c = c_1 \cdot d$ for some c_1. If we divide everything by d we obtain $a_1^2 + b_1^2 = c_1^2$ and $a_1 < b_1 < c_1$ is an arithmetical progression. This shows that any solution (a, c, b) is a multiple of a primitive Pythagorean triple (a_1, c_1, b_1) that is also a solution.

Stage 2. Find the primitive solutions.

Recall that a primitive PT has the form $(p^2 - q^2, p^2 + q^2, 2pq)$ for some $0 < q < p$ satisfying $\gcd(p, q) = 1$.

Since $a_1 < b_1 < c_1$ is an arithmetic progression we have $b_1 = \frac{a_1+c_1}{2}$. At most one of a_1, b_1, c_1 can be even, by primitivity, and it cannot be a_1 or c_1, since $b_1 = \frac{a_1+c_1}{2}$. Thus a_1 is odd. It follows that we cannot have $a_1 = 2pq$, hence $a_1 = p^2 - q^2$. This forces $b_1 = 2pq$ and $c_1 = p^2 + q^2$. Thus,

$$2pq = b_1 = \frac{a_1 + c_1}{2} = p^2.$$

Dividing first and last terms by p yields $2q = p$. Replacing all p's with $2q$ we obtain

- $a_1 = p^2 - q^2 = (2q)^2 - q^2 = 3q^2$.
- $b_1 = 2pq = 2(2q)q = 4q^2$.
- $c_1 = p^2 + q^2 = (2q)^2 + q^2 = 5q^2$.

Thus, $(a_1, c_1, b_1) = (3q^2, 5q^2, 4q^2)$. But this is a primitive triple, so we must have $q = 1$, hence $(a_1, c_1, b_1) = (3, 5, 4)$.

Stage 3. Summary:

The argument shows that, after dividing out the largest common factor from (a, c, b) we obtain $(a_1, c_1, b_1) = (3, 5, 4)$, so the only Pythagorean triples whose entries are an arithmetical progression are the multiples of the triple $(3, 5, 4)$.