The Cayley Representation Theorem, Version 1

The purpose of the theorem

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!)

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example. A 4-variable associative law,

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example. A 4-variable associative law, like

$$(\forall f)(\forall g)(\forall h)(\forall k)((((f \circ g) \circ h) \circ k) = (f \circ (g \circ (h \circ k))),$$

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example. A 4-variable associative law, like

$$(\forall f)(\forall g)(\forall h)(\forall k)((((f \circ g) \circ h) \circ k) = (f \circ (g \circ (h \circ k))),$$

is a consequence of the 3-variable associative law,

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example. A 4-variable associative law, like

$$(\forall f)(\forall g)(\forall h)(\forall k)((((f \circ g) \circ h) \circ k) = (f \circ (g \circ (h \circ k))),$$

is a consequence of the 3-variable associative law, since any structure that satisfies the 3-variable version will also satisfy the 4-variable version.

The purpose of the theorem

Our goal is to convince ourselves that the only **laws** of functional composition (+ identity element!) are the consequences of

1. (Associative Law) $(\forall f)(\forall g)(\forall h)(f \circ (g \circ h) = (f \circ g) \circ h)$,
2. (Unit Laws) $(\forall f)(1 \circ f = f)$ and $(\forall f)(f \circ 1 = f)$.

Here a **law** is a universally quantified equation. A law,

$$L = (\forall \mathbf{x})(s(\mathbf{x}) = t(\mathbf{x})),$$

is a **consequence** of the Associative Law + Unit Laws if every structure that satisfies the Associative Law + Unit Laws also must satisfy L.

Example. A 4-variable associative law, like

$$(\forall f)(\forall g)(\forall h)(\forall k)((((f \circ g) \circ h) \circ k) = (f \circ (g \circ (h \circ k))),$$

is a consequence of the 3-variable associative law, since any structure that satisfies the 3-variable version will also satisfy the 4-variable version. (Proof?)

The theorem

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Before starting the proof, let me describe some special functions $\lambda_m \colon M \to M$. Suppose the multiplication table for $\mathbb{M}$ is

$\circ$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Before starting the proof, let me describe some special functions $\lambda_m \colon M \to M$. Suppose the multiplication table for $\mathbb{M}$ is

$\circ$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

The rows of this 2d-array define function tables for left multiplication operations $\lambda_m \colon M \to M \colon x \mapsto m \circ x$:

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Before starting the proof, let me describe some special functions $\lambda_m \colon M \to M$. Suppose the multiplication table for $\mathbb{M}$ is

$\circ$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

The rows of this 2d-array define function tables for left multiplication operations $\lambda_m \colon M \to M \colon x \mapsto m \circ x$:

x	1	a	b
$\lambda_1(x)$	1	a	b

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Before starting the proof, let me describe some special functions $\lambda_m \colon M \to M$. Suppose the multiplication table for $\mathbb{M}$ is

$\circ$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

The rows of this 2d-array define function tables for left multiplication operations $\lambda_m \colon M \to M \colon x \mapsto m \circ x$:

x	1	a	b
$\lambda_1(x)$	1	a	b

x	1	a	b
$\lambda_a(x)$	a	b	b

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Before starting the proof, let me describe some special functions $\lambda_m : M \to M$. Suppose the multiplication table for $\mathbb{M}$ is

$\circ$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

The rows of this 2d-array define function tables for left multiplication operations $\lambda_m : M \to M : x \mapsto m \circ x$:

x	1	a	b
$\lambda_1(x)$	1	a	b

x	1	a	b
$\lambda_a(x)$	a	b	b

x	1	a	b
$\lambda_b(x)$	b	b	b

The theorem

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof:

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1 \rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1 \rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \text{Funct}(M, M)$.

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \mathrm{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \mathrm{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

Theorem. If $\mathbb{M} = \langle M; \circ, 1 \rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1 \rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \text{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

In symbols, $\lambda_m(x) = m \circ x$;

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \mathrm{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

In symbols, $\lambda_m(x) = m \circ x$; in words, λ_m is 'left multiplication by m'.

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \text{Funct}(M, M)$. This set of functions contains all “left multiplication functions”: for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

In symbols, $\lambda_m(x) = m \circ x$; in words, λ_m is ‘left multiplication by m’.

Claims.

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \mathrm{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

In symbols, $\lambda_m(x) = m \circ x$; in words, λ_m is 'left multiplication by m'.

Claims.

(i) The function $h \colon \mathbb{M} \to \mathbb{F} \colon m \mapsto \lambda_m$ is a homomorphism of monoids, and

The theorem

Theorem. If $\mathbb{M} = \langle M; \circ, 1\rangle$ is a monoid, then $\mathbb{M}$ is embeddable in a function algebra $\mathbb{F} = \langle \mathcal{F}; \circ, 1\rangle$.

Proof: Given $\mathbb{M}$, the function algebra we will use will involve $\mathcal{F} = \mathrm{Funct}(M, M)$. This set of functions contains all "left multiplication functions": for $m \in M$:

$$\lambda_m \colon M \to M \colon x \mapsto m \circ x.$$

In symbols, $\lambda_m(x) = m \circ x$; in words, λ_m is 'left multiplication by m'.

Claims.

(i) The function $h \colon \mathbb{M} \to \mathbb{F} \colon m \mapsto \lambda_m$ is a homomorphism of monoids, and

(ii) h is injective.

Claim (i)

Claim (i)

Claim (i).

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$.

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$,

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of <u>functions</u> in $\mathcal{F}$, so we must show that

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of <u>functions</u> in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x)$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of <u>functions</u> in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities,

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$,

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law.

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second,

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\lambda_{m\circ n}(x)$$

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\lambda_{m\circ n}(x) \quad = (m \circ n) \circ x$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\lambda_{m \circ n}(x) = (m \circ n) \circ x \qquad \text{(Definition of } \lambda_{m \circ n}\text{)}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned} \lambda_{m\circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m\circ n}) \\ &= m \circ (n \circ x) \end{aligned}$$

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned}
\lambda_{m \circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m \circ n}\text{)} \\
&= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M\text{)} \\
&= \lambda_m(\lambda_n(x)). && \text{(Definition of } \lambda_m, \lambda_n\text{)} \\
&= (\lambda_m \circ \lambda_n)(x). && \text{(Definition of composition)}
\end{aligned}$$

In passing from the first line to the second we made use of the Associative Law.

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned} \lambda_{m \circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m \circ n}) \\ &= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}, \text{ since } m, n, x \in M) \end{aligned}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned} \lambda_{m\circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m\circ n}\text{)} \\ &= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M\text{)} \\ &= \lambda_m(\lambda_n(x)). \end{aligned}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned} \lambda_{m \circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m \circ n}) \\ &= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M) \\ &= \lambda_m(\lambda_n(x)). && \text{(Definition of } \lambda_m, \lambda_n) \end{aligned}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned} \lambda_{m\circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m\circ n}) \\ &= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M) \\ &= \lambda_m(\lambda_n(x)). && \text{(Definition of } \lambda_m, \lambda_n) \\ &= (\lambda_m \circ \lambda_n)(x). \end{aligned}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m\circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned}
\lambda_{m\circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m\circ n}) \\
&= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M) \\
&= \lambda_m(\lambda_n(x)). && \text{(Definition of } \lambda_m, \lambda_n) \\
&= (\lambda_m \circ \lambda_n)(x). && \text{(Definition of composition)}
\end{aligned}$$

Claim (i)

Claim (i). The function $h\colon \mathbb{M} \to \mathbb{F}\colon m \mapsto \lambda_m$ is a homomorphism of monoids.

Must show that $h(1) = 1$ and $h(m \circ n) = h(m) \circ h(n)$. These equalities are equalities of functions in $\mathcal{F}$, so we must show that

$$(\forall x)(\lambda_1(x) = x) \quad \text{and} \quad (\forall x)(\lambda_{m \circ n}(x) = (\lambda_m \circ \lambda_n)(x)).$$

For the first of these equalities, for any choice of $x \in M$, $\lambda_1(x) = 1 \circ x = x$ because of the Left Identity Law. For the second, for any choice of $x \in M$,

$$\begin{aligned}
\lambda_{m \circ n}(x) &= (m \circ n) \circ x && \text{(Definition of } \lambda_{m \circ n}) \\
&= m \circ (n \circ x) && \text{(Associative Law in } \mathbb{M}\text{, since } m, n, x \in M) \\
&= \lambda_m(\lambda_n(x)). && \text{(Definition of } \lambda_m, \lambda_n) \\
&= (\lambda_m \circ \lambda_n)(x). && \text{(Definition of composition)}
\end{aligned}$$

In passing from the first line to the second we made use of the Associative Law.

Claim (ii)

Claim (ii)

Claim (ii).

Claim (ii)

Claim (ii). h is injective.

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M.

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$.

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function;

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim.

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim:

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$.

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\lambda_m(1)$$

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\lambda_m(1) \quad = m \circ 1$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\lambda_m(1) \quad = m \circ 1 \qquad \text{(Definition of } \lambda_m\text{)}$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 \quad &\text{(Definition of } \lambda_m) \\ &= m \end{aligned}$$

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \end{aligned}$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \\ &\neq n \end{aligned}$$

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \\ &\neq n && \text{(Initial Assumption)} \end{aligned}$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \\ &\neq n && \text{(Initial Assumption)} \\ &= n \circ 1 && \end{aligned}$$

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \\ &\neq n && \text{(Initial Assumption)} \\ &= n \circ 1 && \text{(Right Identity Law)} \end{aligned}$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned}
\lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\
&= m && \text{(Right Identity Law)} \\
&\neq n && \text{(Initial Assumption)} \\
&= n \circ 1 && \text{(Right Identity Law)} \\
&= \lambda_n(1).
\end{aligned}$$

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$
\begin{aligned}
\lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\
&= m && \text{(Right Identity Law)} \\
&\neq n && \text{(Initial Assumption)} \\
&= n \circ 1 && \text{(Right Identity Law)} \\
&= \lambda_n(1). && \text{(Definition of } \lambda_n\text{)}
\end{aligned}
$$

Claim (ii)

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned}
\lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\
&= m && \text{(Right Identity Law)} \\
&\neq n && \text{(Initial Assumption)} \\
&= n \circ 1 && \text{(Right Identity Law)} \\
&= \lambda_n(1). && \text{(Definition of } \lambda_n\text{)}
\end{aligned}$$

This argument depends on the Right Identity Law.

Claim (ii). h is injective.

We must argue that $h(m) = h(n)$ implies that $m = n$ in M. In other words, we must argue that if $h(m) = \lambda_m(x)$ and $h(n) = \lambda_n(x)$ are equal as functions, then $m = n$. In the contrapositive form, we must show that if $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are NOT the same function; i.e., they disagree on some x.

Subclaim. If $m \neq n$, then $\lambda_m(x)$ and $\lambda_n(x)$ are different functions. In fact, they disagree at the element $x = 1$.

Proof of Subclaim: Assume that $m \neq n$. Then

$$\begin{aligned} \lambda_m(1) &= m \circ 1 && \text{(Definition of } \lambda_m\text{)} \\ &= m && \text{(Right Identity Law)} \\ &\neq n && \text{(Initial Assumption)} \\ &= n \circ 1 && \text{(Right Identity Law)} \\ &= \lambda_n(1). && \text{(Definition of } \lambda_n\text{)} \end{aligned}$$

This argument depends on the Right Identity Law. □

Significance

Significance

A strong version of the Łos-Tarski theorem,

Significance

A strong version of the Łos-Tarski theorem, from model theory,

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h \colon \mathbb{A} \to \mathbb{B}$ is an embedding,

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem,

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds:

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras. Hence, the class of functions algebras satisfies the same universally quantified sentences as the class of monoids.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras. Hence, the class of functions algebras satisfies the same universally quantified sentences as the class of monoids. The universally quantified sentences that hold in all monoids are the Associative Law and the Unit Laws + their consequences

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras. Hence, the class of functions algebras satisfies the same universally quantified sentences as the class of monoids. The universally quantified sentences that hold in all monoids are the Associative Law and the Unit Laws + their consequences (since that is how they are defined).

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras. Hence, the class of functions algebras satisfies the same universally quantified sentences as the class of monoids. The universally quantified sentences that hold in all monoids are the Associative Law and the Unit Laws + their consequences (since that is how they are defined). Hence, the universally quantified sentences that hold in all function algebras are the Associative Law and the Unit Laws + their consequences.

Significance

A strong version of the Łos-Tarski theorem, from model theory, states

Theorem. If $h\colon \mathbb{A} \to \mathbb{B}$ is an embedding, then any universal sentence that holds in $\mathbb{B}$ also holds in $\mathbb{A}$.

If you combine this with the Cayley Representation Theorem, you find that any universally quantified sentence that holds in all function algebras will hold in all monoids. The converse also holds: any universally quantified sentence that holds in all monoids will hold in all functions algebras. Hence, the class of functions algebras satisfies the same universally quantified sentences as the class of monoids. The universally quantified sentences that hold in all monoids are the Associative Law and the Unit Laws + their consequences (since that is how they are defined). Hence, the universally quantified sentences that hold in all function algebras are the Associative Law and the Unit Laws + their consequences.

We have answered the question: *What are the laws of functional composition?*

What if we allow existential quantification?

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular,

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0\rangle$ is not regular.

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0 \rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

   $$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0\rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0 \rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

$$(\exists x)(x \neq 1) \to (\exists e)((e \neq 1) \wedge (e \circ e = e)).$$

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

$$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0 \rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

$$(\exists x)(x \neq 1) \to (\exists e)((e \neq 1) \wedge (e \circ e = e)).$$

   Not all nontrivial monoids have non-identity idempotents,

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

   $$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0\rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

   $$(\exists x)(x \neq 1) \to (\exists e)((e \neq 1) \wedge (e \circ e = e)).$$

   Not all nontrivial monoids have non-identity idempotents, for example $\langle \mathbb{N}; +, 0\rangle$.

What if we allow existential quantification?

If we allow existential quantification, then we can say even more about functional composition!

1. (Function algebras are **regular** monoids)

   $$(\forall f)(\exists g)(f \circ g \circ f = f).$$

   Not all monoids are regular, for example $\langle \mathbb{N}; +, 0\rangle$ is not regular.

2. (Function algebras of more than one element have non-identity **idempotent** elements)

   $$(\exists x)(x \neq 1) \to (\exists e)((e \neq 1) \wedge (e \circ e = e)).$$

   Not all nontrivial monoids have non-identity idempotents, for example $\langle \mathbb{N}; +, 0\rangle$.