ALGEBRA
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define the "*index*" of a subgroup H in a group G.

The **index** of H in G is the cardinality of the set of left cosets of H in G.

2. State Lagrange's Theorem.

Lagrange's Theorem states that if G is a finite group and S is a subgroup of G, then $|S|$ divides $|G|$.

3. Give examples of the following:

(a) A 3×3 Latin square that is not the multiplication table of a group.

a	b	c
c	a	b
b	c	a

Assume that this Latin square is the multiplication table of some group G. Then, because the only element on the diagonal is a, G must satisfy $(\forall x)(x^2 = a)$, and in particular G must satisfy $a^2 = a$. This forces a to be the identity element of G, since the identity element of a group is the only element that is equal to its own square. But now, since $(\forall x)(x^2 = a = 1_G)$ in G, it follows that every nonidentity element of G has order 2. Lagrange's Theorem would then imply that 2 divides $|G| = 3$, which is impossible. This contradiction completes the justification that the above table cannot be the multiplication of a group.

(b) An infinite nonabelian group.

There are many examples, such as:

- the symmetric group S_X where X is an infinite set;
- a matrix group $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, provided $n > 2$;
- the infinite dihedral group $D_\infty = \langle r, f \mid f^2 = 1, fr = r^{-1}f\rangle$.

(c) A group G with a subgroup H such that $[G\colon H] = 3$.

Let $G = C_3 = \langle \{1, r, r^2\}; \cdot, {}^{-1}, 1\rangle$ and let $H = \{1\}$. We have $[G\colon H] = \frac{|G|}{|H|} = \frac{3}{1} = 3$.

4. Suppose that G and H are finite groups and that no prime number divides both $|G|$ and $|H|$. Explain why the only homomorphism from G to H is the constant function that maps every element in G to the identity element of H.

Let $h\colon G \to H$ be a homomorphism. Note that $\mathrm{im}(h)$ is a subgroup of H, hence $d := |\mathrm{im}(h)|$ (> 0) is a divisor of $|H|$ by Lagrange's Theorem. Note that $\mathrm{coim}(h)$ is a quotient of G of cardinality $e := |\mathrm{coim}(h)| = |G/Ker(h)| = [G\colon Ker(h)]$ (> 0), which is a divisor of G according to the equality $|G| = [G\colon Ker(h)] \cdot |Ker(h)|$. By the First Isomorphism Theorem, $\mathrm{coim}(h) \cong \mathrm{im}(h)$, so $e = |\mathrm{coim}(h)| = |\mathrm{im}(h)| = d$, so d $(= e)$ is a divisor of both $|G|$ and $|H|$, hence d divides $\gcd(|G|, |H|)$. The hypothesis of the problem guarantees that $\gcd(|G|, |H|) = 1$, so d divides 1, so $1 = d = |\mathrm{im}(h)|$. This is enough to show that h is a constant function. Since $1_H \in \mathrm{im}(h)$, we get that h is the constant function that maps every element in G to the identity element of H.