HW 9: solution sketches

(1) (Exercise 3.2.4) Show that the permutation group S_n is a semidirect product of $\mathbb{Z}_2$ and the group of even permutations A_n .

The problem should say that S_n is isomorphic to a semidirect product of $\mathbb{Z}_2$ and the group of even permutations A_n. The problem should also say that $n \geq 2$.

It suffices to show that the normal subgroup A_n of S_n has a complementary subgroup of size 2. One such complement is $Q = \langle(1\ 2)\rangle$.

To see that Q is a complement: (i) $Q \cap A_n = \{1\}$ since $Q = \{1, (1\ 2)\}$ and $(1\ 2) \notin A_n$. (ii) $Q \cdot A_n = S_n$, since A_n is a maximal subgroup of S_n (I use the fact that $[S_n : A_n] = 2$ is prime to conclude this), and $Q \cdot A_n$ properly extends A_n (I use the fact that $(1\ 2) \in Q \cdot A_n - A_n$ to conclude this).

(2) (Exercise 3.2.6.) $\mathbb{Z}_4$ has a subgroup isomorphic to $\mathbb{Z}_2$, namely the subgroup generated by [2]. The quotient $\mathbb{Z}_4/\mathbb{Z}_2$ is also isomorphic to $\mathbb{Z}_2$. Nevertheless, $\mathbb{Z}_4$, is not a direct or semidirect product of two copies of $\mathbb{Z}_2$.

Any subgroup of a cyclic group is cyclic and there is a unique subgroup of each order dividing the order of the whole group. Thus, the subgroup lattice of $\mathbb{Z}_n$ is isomorphic to the lattice of divisors of n. If n is a prime power (say $n = p^k$), then the subgroup lattice is a chain (of length $k + 1$).

In particular, the subgroup lattice of $\mathbb{Z}_4$ is a 3-element chain. This is enough to imply that no proper, nontrivial subgroup of $\mathbb{Z}_4$ has a complement, and therefore $\mathbb{Z}_4$ cannot be properly decomposed using the direct product or semidirect product constructions.

(3) Show that if $G \cong A \rtimes B$ and G is abelian, then $G \cong A \times B$.

If $G \cong A \rtimes B$, then G has a normal subgroup isomorphic to A that has a complement isomorphic to B. Since G is abelian, B is also normal in G, so $G \cong A \times B$.