HW 8: solution sketches

(1) Show that the infinite cyclic group $\mathbb{Z}$ cannot factor as $\mathbb{Z} \cong A \times B$ with both A and B containing more than one element.

It suffices to explain why $\mathbb{Z}$ does not have a pair of complementary (normal) subgroups A and B where both subgroups have more than one element.

Assume instead that $A, B \leq \mathbb{Z}$ are complementary subgroups and $m \in A - \{0\}$ and $n \in B - \{0\}$ witness that $A \neq \{0\} \neq B$. The element $mn \in (A \cap B) - \{0\}$ contradicts the property that $A \cap B = \{0\}$ (which is part of what it means for A and B to be complementary).

(2) Find groups so that $A_1 \times A_2 \cong B_1 \times B_2$ with no A_i isomorphic to any B_j.

We can do this by parenthesizing the product $C_2 \times C_2 \times C_2 \times C_2 = C_2^4$ in two different ways:

(1) $A_1 = C_2, A_2 = C_2^3$: $A_1 \times A_2 = C_2 \times (C_2 \times C_2 \times C_2)$.
(2) $B_2 = C_2^2 = B_2$: $B_1 \times B_2 = (C_2 \times C_2) \times (C_2 \times C_2)$.

These choices yield

$$A_1 \times A_2 \cong C_2^4 \cong B_1 \times B_2.$$

Since no A_i has the same size as any B_j, no A_i can be isomorphic to any B_j.

(3) The polar form of a nonzero complex number is $z = r(\cos(\theta) + i\sin(\theta))$ where $r \in \mathbb{R}^+$ is the absolute value of z (i.e., if $z = a + bi$, then $r = |z| = \sqrt{a^2 + b^2}$), and θ is the argument of z (i.e., if $z = a + bi$, then $\tan(\theta) = b/a$). The polar form is useful for understanding the multiplication of complex numbers, since if $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then $z_1 z_2 = (r_1 r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$. ¡p¿ By considering the polar form of a complex number, explain why $\mathbb{C}^\times \cong \mathbb{R}^+ \times T$ where $\mathbb{R}^+$ is the multiplicative group of positive real numbers and T is the multiplicative subgroup of $\mathbb{C}^\times$ consisting of elements on the unit circle.

We need to explain why the multiplicative group $\mathbb{R}^+$ is a complement to T.

- $\mathbb{R}^+ \vee T = \mathbb{R}^+ T = \mathbb{C}^\times$ by the polar form representation.
- $\mathbb{R}^+ \wedge T = \mathbb{R}^+ \cap T = \{1\}$ because the only positive real number z on the complex unit circle is $z = 1$.