**HW 7: solution sketches**

(1) (Exercise 2.6.5.) What are the conjugacy classes in the dihedral group D5 ?

If we conjugate a rotation $r^i$ by another rotation $r^j$ it remains the same: $r^j r^i r^{-j} = r^i$. If we conjugate $r^i$ by any flip $fr^j$ it reduces to $(fr^j)(r^i)(r^{-j}f^{-1}) = fr^i f^{-1} = r^{-i}$. Thus, the conjugacy class of $r^i$ is $\{r^i, r^{-i}\}$. This yields conjugacy classes $\{1\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}$. The fact that $r^{3i} f r^{-3i} = r^{6i} f = r^i f$ shows that $f$ is conjugate to any other flip, $r^i f$, so the full list of conjugacy classes is

$$\{1\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \{f, rf, r^2 f, r^3 f, r^4 f\}.$$

(2) (Exercise 2.7.4.) Suppose $G$ is a finite group. Let $N$ be a normal subgroup of $G$ and $A$ an arbitrary subgroup. Verify that $|AN| = \frac{|A||N|}{|A \cap N|}$.

By following the hint, we can prove this statement even if $N$ is not normal in $G$.

**Stage 1.** For any $g \in AN$ there exist $a \in A$ and $n \in N$ such that $p(a, n) = an = g$, so $p$ is surjective.

**Stage 2.** $p^{-1}(1) = \{(a, n) \in A \times N \mid an = 1\}$. This set consists of those pairs $(a, a^{-1})$ where $a \in A$ and $a^{-1} \in N$, equivalently those pairs $(a, a^{-1})$ where $a \in A \cap N$. The number of these pairs is $|A \cap N|$, since the function $A \cap N \to p^{-1}(1)\colon a \mapsto (a, a^{-1})$ is a bijection.

**Stage 3.** Now choose any element $g = xy \in AN$ where $x \in A, y \in N$. The function

$$p^{-1}(1) \to p^{-1}(g)\colon (a, n) \mapsto (xa, ny)$$

is a bijection with inverse

$$p^{-1}(g) \to p^{-1}(1)\colon (a, n) \mapsto (x^{-1}a, by^{-1}).$$

This shows that all fibers of $p$ have the same size, so the kernel of $p$ is uniform.

$\ker(p)$ is a uniform partition of the set $A \times B$ that has $|\mathrm{im}(p)| = |AN|$ cells, each of size $|p^{-1}(1)| = |A \cap B|$, so

$$|AN| = \frac{|A \times B|}{|A \cap B|} = \frac{|A||B|}{|A \cap B|}.$$

(3) (Exercise 2.7.6(b).) Recall that an automorphism of a group $G$ is a group isomorphism from $G$ to $G$. Denote the set of all automorphisms of $G$ by $\mathrm{Aut}(G)$.

(b) Recall that for each $g \in G$, the map $c_g\colon G \to G$ defined by $c_g(x) = gxg^{-1}$ is an element of $\mathrm{Aut}(G)$. Show that the map $c\colon g \mapsto c_g$ is a homomorphism from $G$ to $\mathrm{Aut}(G)$.

It suffices to verify that $c$ preserves multiplication, i.e., that $(\forall x)(c(gh)(x) = c(g) \circ c(h)(x))$ holds. We check:

$$c(gh)(x) = c_{gh}(x) = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = c_g(c_h(x)) = (c(g) \circ c(h))(x).$$