HW 6: solution sketches

(1) Let G be a finite group and assume that $H, K \leq G$ are subgroups. Explain why
- $|H \cap K|$ divides $\gcd(|H|, |K|)$.
- $|\langle H \cup K\rangle|$ is divisible by $\operatorname{lcm}(|H|, |K|)$.

$H \cap K$ is a subgroup of both H and K, so Lagrange's Theorem guarantees that $|H \cap K|$ divides both $|H|$ and $|K|$. By the definition of gcd, $|H \cap K|$ divides $\gcd(|H|, |K|)$.

Both H and K are subgroups of $\langle H \cup K\rangle$, so Lagrange's Theorem guarantees that both $|H|$ and $|K|$ divide $\langle H \cup K\rangle$. By the definition of lcm, $\operatorname{lcm}(|H|, |K|)$ divides $|\langle H \cup K\rangle|$.

(2) Show that the normality relation is not transitive by showing that, in D_4, we have $\langle f\rangle \triangleleft \langle r^2, f\rangle$ and $\langle r^2, f\rangle \triangleleft D_4$, but not $\langle f\rangle \triangleleft D_4$.

By HW5(3) (Exercise 2.5.8) we know that if $[G\colon H] = 2$, then $H \triangleleft G$. Since $|\langle f\rangle| = 2, |\langle r^2, f\rangle| = 4$, and $|D_4| = 8$, we must have $[\langle r^2, f\rangle\colon \langle f\rangle] = 2$ and $[D_4\colon \langle r^2, f\rangle] = 2$, so $\langle r^2, f\rangle \triangleleft \langle r^2, f\rangle \triangleleft D_4$. It remains to show that $\langle f\rangle \not\triangleleft D_4$.

The subgroup $H = \langle f\rangle$ of D_4 is not normal in D_4, since the left coset $rH = \{r, rf\}$ is not equal to the right coset $Hr = \{r, fr\} = \{r, r^{-1}f\}$.

(3) Assume that G is a finite group, $H \leq G$ is a subgroup, and $g \in G$. Show that gHg^{-1} is a subgroup of G and $|H| = |gHg^{-1}|$. (Hint: First show that the function $x \mapsto gxg^{-1}$ is an automorphism of G.)

Let's show verify that for any group G and any $g \in G$ the function

$$\gamma_g\colon G \to G\colon x \mapsto gxg^{-1}$$

is an automorphism of G. The verification is this line:

$$\gamma_g(xy) = gxyg^{-1} = gx\underline{g^{-1}g}yg^{-1} = \gamma_g(x)\gamma_g(y).$$

Next, we know that the image of a subgroup H of G under a homomorphism is a subgroup, so $gHg^{-1} = \gamma_g(H) \leq G$.