

## HW 5: solution sketches

- (1) (Exercise 2.5.4) Consider the group  $S_3$ .
- (a) Find all the left cosets and all the right cosets of the subgroup  $H = \{e, (1\ 2)\}$  of  $S_3$ , and observe that not every left coset is also a right coset.
  - (b) Find all the left cosets and all the right cosets of the subgroup  $K = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$  of  $S_3$ , and observe that every left coset is also a right coset.

### Left cosets of $H$ .

$$\begin{aligned}eH &= (1\ 2)H = \{e, (1\ 2)\}, \\(1\ 2\ 3)H &= (1\ 3)H = \{(1\ 2\ 3), (1\ 3)\}, \\(1\ 3\ 2)H &= (2\ 3)H = \{(1\ 3\ 2), (2\ 3)\}.\end{aligned}$$

### Right cosets of $H$ .

$$\begin{aligned}He &= H(1\ 2) = \{e, (1\ 2)\}, \\H(1\ 2\ 3) &= H(2\ 3) = \{(1\ 2\ 3), (2\ 3)\}, \\H(1\ 3\ 2) &= H(1\ 3) = \{(1\ 3\ 2), (1\ 3)\}.\end{aligned}$$

### Left cosets of $K$ .

$$\begin{aligned}eK &= (1\ 2\ 3)K = (1\ 3\ 2)K = \{e, (1\ 2\ 3), (1\ 3\ 2)\}, \\(1\ 2)K &= (1\ 3)K = (2\ 3)K = \{(1\ 2), (1\ 3), (2\ 3)\},\end{aligned}$$

### Right cosets of $K$ .

$$\begin{aligned}Ke &= K(1\ 2\ 3) = K(1\ 3\ 2) = \{e, (1\ 2\ 3), (1\ 3\ 2)\}, \\K(1\ 2) &= K(1\ 3) = K(2\ 3) = \{(1\ 2), (1\ 3), (2\ 3)\},\end{aligned}$$

- (2) (Exercise 2.5.6) Let  $H$  be a subgroup of a group  $G$ . Show that  $aH \mapsto Ha^{-1}$  defines a bijection between left cosets of  $H$  in  $G$  and right cosets of  $H$  in  $G$ . (The index of a subgroup was defined in terms of left cosets, but this observation shows that we get the same notion using right cosets instead.)

Inverting every element of the left coset  $aH = \{ah \mid h \in H\}$  yields  $(aH)^{-1} = \{(ah)^{-1} = h^{-1}a^{-1} \mid h \in H\}$ . As  $h$  runs through the elements of  $H$ ,  $h^{-1}$  also runs through the elements of  $H$ , since inversion is bijection and  $H$  is closed under inversion. Thus,

$$(aH)^{-1} = \{(ah)^{-1} = h^{-1}a^{-1} \mid h \in H\} = \{ha^{-1} \mid h \in H\} = Ha^{-1}.$$

This shows that inversion on  $G$  is a map from left cosets of  $H$  to right cosets of  $H$ . Similarly, inversion on  $G$  is a map from right cosets of  $H$  to left cosets of  $H$ ; in fact, inversion is its own inverse:  $((aH)^{-1})^{-1} = (Ha)^{-1} = aH$ . This shows that inversion is a bijection from left cosets to right cosets.

- (3) (Exercise 2.5.8) Suppose  $N$  is a subgroup of a group  $G$  and  $[G : N] = 2$ . Show that  $N$  is normal using the criterion of Exercise 2.5.7.

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Assume that  $[G: N] = 2$ . If  $a \in G - N$ , then  $aN \neq N$  since  $a \in aN - N$ . The partition of  $G$  into two left cosets of  $N$  must be

$$G/N = \{N, aN\} = \{N, G - N\}.$$

The same argument shows that the partition  $G$  into two right cosets of  $N$  must be  $N \backslash G = \{G, G - N\}$ . Since all left cosets of  $N$  are right cosets,  $N$  is normal.