HW 3: solution sketches

(1) Suppose that $\varphi\colon G \to H$ is an isomorphism of groups. Show that G is abelian if and only if H is abelian.

It suffices to prove that if $\varphi\colon G \to H$ is an isomorphism of groups, then the property "G is abelian" **implies** the property "H is abelian". The **reverse implication** can be derived by a similar argument applied to the inverse isomorphism $\varphi^{-1}\colon H \to G$.

Assume that $\varphi\colon G \to H$ is an isomorphism and that G is abelian. Choose arbitrary $h_1, h_2 \in H$. Our goal is to prove that $h_1 \cdot h_2 = h_2 \cdot h_1$. We calculate

$$h_1 \overset{H}{\cdot} h_2 = \varphi(\varphi^{-1}(h_1) \overset{G}{\cdot} \varphi^{-1}(h_2)) = \varphi(\varphi^{-1}(h_2) \overset{G}{\cdot} \varphi^{-1}(h_1)) = h_2 \overset{H}{\cdot} h_1.$$

The middle equality holds because multiplication in G is commutative.

(2) Generalizing Exercise 2.1.10, show that any group with an even number of elements must have a nonidentity element whose square is the identity, that is, a nonidentity element that is its own inverse.

Let G be a group with an even number of elements. Define an equivalence relation on G by the condition that $x \equiv y$ iff (i) $x = y$ or (ii) $x = y^{-1}$. (The relation $\equiv$ defined by (i) and (ii) is **reflexive** by definition, and **symmetric** by the fact that $(x^{-1})^{-1} = x$. To verify **transitivity**, there are a few cases to consider, but the only nonobvious case to check is: $x \equiv y$ because (ii) $x = y^{-1}$ and $y \equiv z$ because (ii) $y = z^{-1}$. In this case, $x \equiv z$ because (i) $x = (z^{-1})^{-1} = z$.)

The equivalence classes of the relation $\equiv$ have size s for some choices $1 \leq s \leq 2$. Since $\{1\} = (1/\equiv)$ is a $\equiv$-class of size 1, and the sum of the sizes of all $\equiv$-classes is $|G|$ = an even number, there must be some other class $\{g\} = (g/\equiv)$ of size 1 where $g \neq 1$. Thus, since $g^{-1} \equiv g \in (g/\equiv) = \{g\}$, we get $g^{-1} \in \{g\}$, or $g^{-1} = g$ for some $g \neq 1$.

(3) Show that the following conditions are equivalent for a group G:
   (a) G is abelian.
   (d) For all $a, b \in G$, $a^2b^2 = (ab)^2$.
   (e) For all $a, b \in G$ and natural numbers n, $(ab)^n = a^nb^n$.

We prove the equivalence of the statements by proving three implications in the order (e)⇒(d)⇒(a)⇒(e).

(e)⇒(d) If (e) holds for all n, then it holds for $n = 2$. This is statement (d).

(d)⇒(a) Assume that (d) holds. For any $a, b \in G$ we have $aabb = a^2b^2 = (ab)^2 = abab$. Cancelling one a from the left and one b from the right of the equality $aabb = abab$ yields $ab = ba$. This conclusion holds for all $a, b \in G$, so (a) holds.

(a)⇒(e) Here we must argue that if G is abelian, then $(ab)^n = a^n b^n$. We argue by induction on n with Base Case $n = 1$.

$$\begin{aligned} (ab)^1 &= ab && \text{(Defn of Exp, Initial Condition)} \\ &= a^1 b^1 && \text{(Defn of Exp, Initial Condition)} \end{aligned}$$

For the Inductive Step, assume as an Inductive Hypothesis that $(ab)^k = a^k b^k$ for some $k \geq 1$.

$$\begin{aligned} (ab)^{k+1} &= (ab)^k (ab) && \text{(Recurrence Relation, Exp)} \\ &= (a^k \underline{b^k})(\underline{a}b) && \text{(Inductive Hypothesis)} \\ &= (a^k \underline{a})(\underline{b^k} b) && \text{(Associative \& Commutative Laws)} \\ &= a^{k+1} b^{k+1} && \text{(Recurrence Relation, Exp)} \end{aligned}$$

This completes an inductive proof that (e) holds.