HW 2: solution sketches

(1) In an earlier HW assignment, you determined the number of 2-element algebras of the form $\langle\{0,1\};\star\rangle$ where $\mathbf{arity}(\star) = 2$. You also determined the number of isomorphism types of such algebras. Let's continue that investigation by answering the following additional question: How many 2-element algebras of the form $\langle\{0,1\};\star\rangle$ have an identity element? First give the number of such algebras up to equality, then state the number of such algebras up to isomorphism.

If the identity element is 0, then the table for $\star$ must be

$\star$	0	1
0	0	1
1	1	0

or

$\star$	0	1
0	0	1
1	1	1

If the identity element is 1, then the table for $\star$ must be

$\star$	0	1
0	1	0
1	0	1

or

$\star$	0	1
0	0	0
1	0	1

These are four different tables, so the answer to the first part of the question is 4. However, the second pair of tables produce algebras isomorphic to the first pair, so the number of isomorphism types is 2. (More detail: in the first two tables, where 0 is an identity element, one of them describes a group multiplication and the other does not. Therefore the tables in the first pair yield nonisomorphic algebras. The second pair of tables differ from the first pair by swapping the roles of 0 and 1, so the swap map will produce the necessary isomorphisms.)

(2) Show that if some algebra $\mathbb{A} = \langle A;\star\rangle$ has an identity element for $\star$, then the identity element for $\star$ is unique.

Assume that $a, b \in A$ are both identity elements for $\star$. Then

$$\begin{aligned} a &= a \star b && (b \text{ is a right identity element}) \\ &= b && (a \text{ is a left identity element}) \end{aligned}$$

(3) Suppose that $\mathbb{A} = \langle A;\star,1\rangle$ is an algebra with one binary operation $\star$ and one zeroary operation 1. Assume that 1 is an identity element of $\mathbb{A}$ with respect to $\star$. If $a \in A$, then an **inverse** to a with respect to $\star$ is an element $b \in A$ such that $a \star b = 1$ and $b \star a = 1$.

(a) Show that if $\star$ is an associative operation, then any $a \in A$ can have at most one inverse.

(b) The purpose of this example is to show that $a \in A$ may have more than one inverse if the multiplication $\star$ is not associative. Give an example of a 3-element algebra $\mathbb{A} = \langle \{1, a, b\}; \star, 1 \rangle$ where (i) 1 is an identity element for $\mathbb{A}$, (ii) every element of $\mathbb{A}$ has an inverse with respect to $\star$, but (iii) inverses are not unique in $\mathbb{A}$. (To "Give an example" it suffices to write down a table for $\star$.)

(a) Choose $b \in A$ and assume that $a, c \in A$ are both inverses of b. Then

$$a = a \star 1 = a \star (b \star c) = (a \star b) \star c = 1 \star c = c.$$

This shows that any two inverses of b are equal.

(b) The only example is the algebra whose multiplication table is:

$\star$	1	a	b
1	1	a	b
a	a	1	1
b	b	1	1

Inverses are not unique, since a and b are both inverses for a (and b).