HW 1: solution sketches

(1) (a) How many algebras are there of the form $\langle\{0,1\};\star\rangle$ if $\mathbf{arity}(\star)=2$? (You need to count how many different tables for $\star$ are possible.)

(b) How many *isomorphism types* of algebras are there of the form $\langle\{0,1\};\star\rangle$? (Count algebras as in part (a), but discard isomorphic copies.)

(a) The possible tables for $\star$ all have the form

$\star$	0	1
0	a	b
1	c	d

where a, b, c and d may chosen independently from the set $\{0,1\}$. There are $2\times 2\times 2\times 2 = 16$ choices for $(a,b,c,d)\in\{0,1\}^4$, so there are 16 possible algebras.

(b) An isomorphism $h\colon \mathbb{A}\to\mathbb{B}$ between algebras with universes $\{0,1\}$ is a bijection $h\colon\{0,1\}\to\{0,1\}$ which preserves the operations. There are two possible bijections $h\colon\{0,1\}\to\{0,1\}$: the identity function $h(x)=x$ on $\{0,1\}$ and the "swap" $h(x)=\neg x$ (Boolean notation). For h to be an isomorphism between two **different** algebras with universe $\{0,1\}$, it must be the swap. If h is the swap, $h\colon\mathbb{A}\to\mathbb{B}$ is an isomorphism, and the $\star$-table for $\mathbb{A}$ is

$\star$	0	1
0	a	b
1	c	d

,

then the $\star$-table for $\mathbb{B}$ must be

$\star$	0	1
0	$\neg d$	$\neg c$
1	$\neg b$	$\neg a$

.

(This table was computed by the condition that $x\star^{\mathbb{B}} y = \neg(\neg x\star^{\mathbb{A}}\neg y)$.) Call the unique $\mathbb{B}$ that has its $\star$-table computed from the $\star$-table of $\mathbb{A}$ in this way the "dual" of $\mathbb{A}$. Any algebra can be isomorphic only to itself (using $h(x)=x$) and to its dual (using $h(x)=\neg x$).

Usually, the dual of a 2-element algebra will be different from the original, but not always. Call $\mathbb{A}$ "self-dual" if it is equal to its dual. $\mathbb{A}$ is self-dual exactly when its table agrees with the table for its dual, which happens exactly when the $\star$-table for $\mathbb{A}$ has the form:

$\star$	0	1
0	e	f
1	$\neg f$	$\neg e$

.

There are four tables like this (i.e., four choices for $(e, f) \in \{0, 1\}$.)

Altogether, this means that there are 4 self-dual algebras (each of which determines a single isomorphism type of 2-element algebra in this signature), while the remaining $16 - 4 = 12$ algebras are partitioned into dual pairs (which determine $12/2 = 6$ more isomorphism types). This yields $4 + 6 = \mathbf{10}$ isomorphism types of 2-element algebras in this signature.

(2) Suppose that a is an identity element for $+$ in $\langle\{a, b, c\}; +\rangle$, where $\mathbf{arity}(+) = 2$. (This means that $a + x = x = x + a$ for every x.) How many possibilities are there for such an algebra? How many possibilities if $+$ is a commutative operation with identity element a? ("Commutative" means that $x + y = y + x$ for every x and y.)

The answers to the two questions are $3^4 = 81$ and $3^3 = 27$, as we now explain. The possible $+$-tables for the first question have the form

$+$	a	b	c
a	a	b	c
b	b	w	x
c	c	y	z

where $(w, x, y, z) \in \{a, b, c\}^4$ may be chosen arbitrarily. There are $|\{a, b, c\}^4| = 3^4 = 81$ such choices.

For $+$ to be commutative, it is necessary and sufficient that $x = y$ in the above explanation. In this case, the $+$-table has the form

$+$	a	b	c
a	a	b	c
b	b	w	x
c	c	x	z

The number of choices for $(w, x, z) \in \{a, b, c\}^3$ is $3^3 = 27$.

(3) Let $\mathbb{Z} = \langle Z; +, -, 0\rangle$. Show that if $h\colon \mathbb{Z} \to \mathbb{Z}$ is a homomorphism, and $h(1) = a$, then $h(n) = an$ for every $n \in Z$. (Hint: First, use induction to prove it for positive n.)

We first prove that $h(n) = an$ for positive n by induction on n:

(Base Case: $n = 1$)

$$\begin{aligned} h(1) &= a && \text{(Definition of } h\text{)} \\ &= a \cdot 1 && \text{(1 is a multiplicative unit)} \end{aligned}$$

(Inductive Step: Assume true for $n = k$ when $k \geq 1$, prove true for $n = k + 1$)

$$\begin{aligned} h(k+1) &= h(k) + h(1) && (h \text{ is a homomorphism}) \\ &= ka + a && \text{(Inductive Hypothesis+Assumption)} \\ &= (k+1)a && \text{(Distributive Law)} \end{aligned}$$

Now, we prove it for $-n$ when $n > 0$. Since h is a homomorphism, $h(-n) = -h(n) = -(an) = a(-n)$.

Finally, we prove it for $n = 0$. Since 0 is a zeroary operation and h is a homomorphism, $h(0) = 0$. Hence $h(0) = 0 = a0$.

To summarize, we have shown that $h(n) = an$ for all values of n that are positive, negative, or zero, so the statement is true.