HW 10: solution sketches

(1) (Exercise 3.6.14) How many abelian groups are there of order p^5q^4, where p and q are distinct primes?

According to the formula developed in class, the number is $\pi(5) \cdot \pi(4) = 7 \cdot 5 = 35$, where $\pi(n)$ denotes the number of partition types of an n-element set.

(2) A number n is square free if it is not divisible by m^2 for any integer $m > 1$. (So, $6, 10, 15$ are square free, but $9, 12, 18$ are not.) If n is square free, what can you say about the number of isomorphism types of abelian groups of order n?

If $n = p_1 \cdot p_2 \cdots p_k = p_1^1 \cdot p_2^1 \cdots p_k^1$, then the number of isomorphism types is $\pi(1) \cdot \pi(1) \cdots \pi(1) = 1$. (This shows that there is a unique isomorphism type of abelian group of order n if n is square free. That group must be C_n.)

(3) Assume that G is a finite abelian p-group. What is the relationship between the number of elementary divisors of G and the number of elements of order p in G?

Suppose that the p-group G has k elementary divisors. Then $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_k}}$ for some choice of exponents $e_i > 0$. An element $(a_1, \ldots, a_k)$ from this product satisfies $x^p = 1$ if an only if each coordinate satisfies it. There are p solutions to $x^p = 1$ in each coordinate, so there are p^k solutions in G. This set of solutions is exactly the set of elements of order p in G augmented by the identity element, so the number of elements of order exactly p must be $p^k - 1$. The relationship is: k elementary divisors $\longleftrightarrow$ $p^k - 1$ elements of order p.