Counting surjective functions

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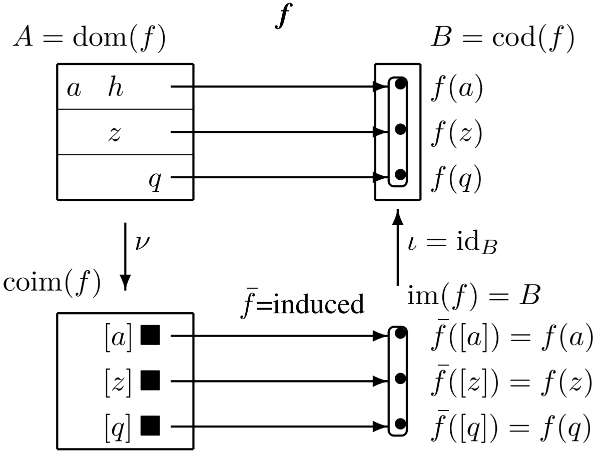

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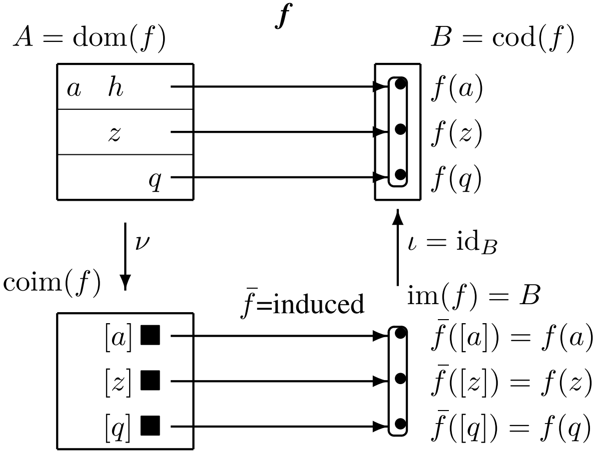

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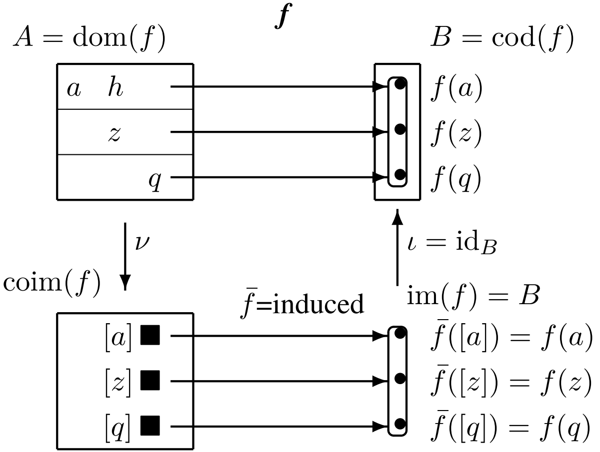

The function f is completely determined by its coimage and its induced function $\bar{f}$. Hence # surjective f's = (# coim's) × (# $\bar{f}$'s).

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Hence, the number of surjective functions $f\colon A \to B$, $|A| = n$, and $|B| = k$, is

$$(\#k\text{-element partitions of } A) \times k!$$

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There are many parallels between $C(n, k)$ and $S(n, k)$.

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□

Binomial-type theorems

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$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$,
where $x^{\underline{k}} = (x)_k = x(x-1)\cdots(x-(k-1))$.

Table of Stirling numbers of the second kind

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$n \backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	0	0	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Each row is a unimodal sequence with maximum occurring for one or two consecutive values around $k \approx \frac{n}{\ln(n)}$.

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4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Each row is a unimodal sequence with maximum occurring for one or two consecutive values around $k \approx \frac{n}{\ln(n)}$.

The n row sum is denoted B_n and is called the nth **Bell number**.

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