Propositional logic

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- W = "The ground is wet".

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R	W	$R \wedge W$
0	0	0
0	1	0
1	0	0
1	1	1

R	W	$R \vee W$
0	0	0
0	1	1
1	0	1
1	1	1

R	$\neg R$
0	1
1	0

R	W	$R \rightarrow W$
0	0	1
0	1	1
1	0	0
1	1	1

R	W	$R \leftrightarrow W$
0	0	1
0	1	0
1	0	0
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R	W	$R \oplus W = R \veebar W$
0	0	0
0	1	1
1	0	1
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Write the truth table for $P := R \lor (((A \land B) \to (\neg R)))$.

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A	B	R	$A \wedge B$	$\neg R$	$((A \wedge B) \to (\neg R))$	P
0	0	0	0	1	1	1
0	0	1	0	0	1	1
0	1	0	0	1	1	1
0	1	1	0	0	1	1
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This proposition P is a **tautology**, because it assumes the value “true” under any truth assignment to the propositional variables.

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This proposition P is a **tautology**, because it assumes the value "true" under any truth assignment to the propositional variables. This means that P is true because of its logical structure alone, and not because of the truth values of its variables.

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(Write $P \equiv Q$.)

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- ($\to$ is redundant) $(P \to Q) \equiv (\neg P) \vee Q$

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Examples.

- $H \to C$ is logically equivalent to $(\neg C) \to (\neg H)$.
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Logical implication, logical equivalence, logical independence

If $P \to Q$ is a tautology, we say that P **logically implies** Q. P is **logically equivalent** to Q (denoted $P \equiv Q$) if any of the following hold:

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Direct implication, contrapositive, converse, inverse

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0	0	1	1	1	1	1	1
0	1	1	0	1	1	0	0
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Example. Compare an example statement to its inverse:

$$\text{“If } \underbrace{\text{we adopt the new policy}}_{H}\text{, then } \underbrace{\text{things will get better}}_{C}\text{.”}$$

$$\text{“If } \underbrace{\text{we do not adopt the new policy}}_{\neg H}\text{, then } \underbrace{\text{things will not get better}}_{\neg C}\text{.”}$$

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The truth table of a monomial has exactly one row whose value is $T = 1$:

A	B	C	D	$(\neg A) \wedge B \wedge C \wedge (\neg D)$
0	0	0	0	0
				$\vdots$
0	1	1	0	1
				$\vdots$
1	1	1	1	0

The monomial $(\neg A) \wedge B \wedge C \wedge (\neg D)$ assumes value 1 iff $A = 0, B = 1, C = 1, D = 0$.

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Theorem. Every propositional formula is logically equivalent to a formula in DNF = $\bigvee(\wedge \pm \text{variables})$.

Corollary. The symbols $\wedge, \vee, \neg$ are a "complete" set of logical connectives, in the sense that any proposition is logically equivalent to one expressed with $\{\wedge, \vee, \neg\}$ + propositional variables.

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If the set of propositional variables to be considered is $\{A, B, C\}$, then the DNF for A is

$$(A \wedge (\neg B) \wedge (\neg C)) \vee (A \wedge (\neg B) \wedge C) \vee (A \wedge B \wedge (\neg C)) \vee (A \wedge B \wedge C)$$

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0	0	1	1
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0	1	1	1
1	0	0	**0**
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