Theorems and their proofs!

- A **Theorem** is a declarative statement that has a *proof*.
- A **Proof** of a theorem T is a finite sequence of statements $S_1, \ldots, S_k = T$ such that
  - each statement follows from earlier statements via an accepted *law of deduction*.
  - the last statement in the proof is T.
- A **Law of Deduction** is rule for drawing a *conclusion* from a set of *premises*.

Examples of laws of deduction.

(1) (Modus Ponens, or "the mode that affirms") Written $\dfrac{(P \to Q), P}{Q}$. It means that if, within a proof, you see

$$S_1, S_1, \ldots, P, \ldots, (P \to Q), \ldots, S_m, \quad \text{or} \quad S_1, S_1, \ldots, (P \to Q), \ldots, P, \ldots, S_m,$$

then you can continue the proof

$$S_1, S_1, \ldots, P, \ldots, (P \to Q), \ldots, S_m, Q \quad \text{or} \quad S_1, S_1, \ldots, (P \to Q), \ldots, P, \ldots, S_m, Q.$$

(2) (Modus Tollens, or "the mode that denies") Written $\dfrac{(P \to Q), \neg Q}{\neg P}$.

(3) (Axioms) Written $\dfrac{}{A}$.

Topics for discussion.

(1) How do we select axioms? rules of deduction?
(2) What is the role of the hypotheses of a theorem?
(3) What is the difference between formal proof and informal proof?
(4) Are all provable statements true? Are all true statements provable? (Soundness, completeness, decidability, Gödel's Completeness Theorem.)
(5) What are some common proof strategies?
  (a) Direct proof.
  (b) Proof by contradiction.
  (c) Proof of the contrapositive.
  (d) Read "Tactics for logical reasoning", subsection 4.8.2.
(6) How can we use propositional tautologies when constructing proofs?

Practice: Prove the theorems directly, by contraposition, and by contradiction (3 proofs each):

Thm. If $x > 0$, then $x + 1 > 0$.

Thm. If $0 < x < 1$, then $x^2 < x$.

Thm. If n is even and $n = k^2$, then k is even.