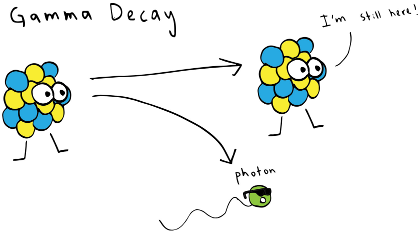
Truth versus Provability



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Answer. Axioms are statements that follow from the empty collection of preceding statements!
So, a "proof system" typically specifies its axioms and also the accepted rules of deduction.

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Why do we believe that Modus Ponens is valid?

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0	0	1	0	1
0	1	1	0	1
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Another way to think about this is: at the first-order level, every statement has a proof or a counterexample.

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Then $\Sigma \models Q$, but $\Sigma \not\vdash Q$ for any proof system requiring finite-length proofs.

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Assume that $0 < x < 1$. Since $0 < x$, multiplication by x preserves inequalities.

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